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On second quantization on noncommutative spaces with twisted symmetries

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Abstract

By the application of the general twist-induced $\star$-deformation procedure we translate second quantization of a system of bosons/fermions on a symmetric spacetime into a noncommutative language. The procedure deforms, in a coordinated way, the spacetime algebra and its symmetries, the wave-mechanical description of a system of n bosons/fermions, the algebra of creation and annihilation operators and also the commutation relations of the latter with functions of spacetime; our key requirement is the mode-decomposition independence of the quantum field. In a minimalistic view, the use of noncommutative coordinates can be seen just as a way to better express non-local interactions of a special kind. In a non-conservative one, we obtain a closed, covariant framework for quantum field theory (QFT) on the corresponding noncommutative spacetime consistent with quantum mechanical axioms and Bose–Fermi statistics. One distinguishing feature is that the field commutation relations remain of the type 'field (anti)commutator=a distribution'. We illustrate the results by choosing as examples interacting non-relativistic and free relativistic QFT on Moyal space(time)s.

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Contents

1. Introduction

The idea of quantum field theory (QFT) on noncommutative spacetime was first proposed by Heisenberg[1], but has been investigated more intensively only in the last 15 years. Motivations range from its appearance in string theory (an effective description in the presence of a $D3$-brane with a large B-field is provided [54] by a non-local Yang–Mills action obtained by replacing the pointwise product by a $\star$-product) to more fundamental ones, like the search [20] for a proper framework where the principles of quantum mechanics and general relativity can be conciliated; such a framework could also yield as a bonus an intrinsic regularization mechanism of ultraviolet divergences in QFT—another important motivation (Heisenberg's original one). Trying to mimic QFT on commutative spacetime, it is desirable to reproduce as many equivalent approaches as possible. However, we believe that their legitimation should come from the equivalence to some operator approach admitting a proper quantum mechanical interpretation. In fact on Moyal spaces, the non-equivalence [6] of the naive Euclidean and Minkowski formulations of relativistic QFT, or of the path-integral [26] and operator approaches to quantization, may be the source [7] of complications such as non-unitarity after naive Wick back-rotation [33], violation of causality [13, 53] even at a large distance, mixing of UV and IR divergences [46] and subsequent non-renormalizability [18], and the need for translation of non-invariant counterterms to recover renormalizability [37].

Historically, second quantization played a crucial role in the foundation of QFT as a bottom-up approach from the wave-mechanical description of a system of n identical quantum particles. For instance, the non-relativistic field operator of a zero-spin particle (in the Schrödinger picture) and its Hermitian conjugate are introduced by

$$\varphi(\mathrm{x}) := \varphi_i(\mathrm{x})a^i, \qquad \varphi^*(\mathrm{x}) = \overline{\varphi_i(\mathrm{x})}a_i^+ \qquad \text{(infinite sum over } i\text{)}, \tag{1}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of a suitable subspace $\mathcal{H}$ dense in the one-particle Hilbert space and φ_i, a_i^+, a^i are the wavefunction, creation and annihilation operators associated with e_i, respectively. It is tempting to *adopt the second quantization approach also on noncommutative spaces*. The main motivation is the wish to start from the particle interpretation of quantum fields and their Bose–Fermi statistics[2] in generic (in particular, 3+1) dimensions. In this paper we adopt the second quantization approach on noncommutative spaces, using a *twist* [21] to deform in a coordinated way space(time), its symmetries and all objects transforming

[1] Heisenberg proposed it in a letter to Peierls [40] to solve the problem of divergent integrals in relativistic QFT. The idea propagated via Pauli to Oppenheimer. In 1947, Snyder, a student of Oppenheimer, published the first concrete proposal of a quantum theory built on a noncommutative space [55]. The idea was put aside after the first successes of renormalization in QED.

[2] Changing the statistics, i.e. the rule to compute the number of allowed states of n-particle systems, would have dramatic physical consequences: even a very small violation of the Pauli exclusion principle would e.g. appreciably modify the behaviour of multi-fermion systems and affect the properties of matter, first of all its stability, especially under extreme conditions (e.g. in neutron stars).

under space(time) transformations; this has the advantage of restoring all the undeformed symmetries in terms of noncocommutative Hopf algebras.

Let us summarize the main conceptual steps. A rather general way to deform an algebra is by deformation quantization [8]. This means keeping the vector space, but deforming the original product $\cdot$ into a new one $\star$. On the space of functions on a manifold X, $f \star h$ can be defined by applying to $f \otimes h$ first a suitable bi-pseudodifferential operator $\overline{\mathcal{F}}$ (reducing to 1 when the deformation parameter λ vanishes) and then the pointwise multiplication $\cdot$. If one replaces all $\cdot$ by $\star$'s in an equation of motion, e.g. in the Schrödinger equation on $X = \mathbb{R}^3$ of a particle with the electric charge q:

$$\mathsf{H}_\star^{(1)}\psi(\mathrm{x}) = \mathrm{i}\hbar\partial_t\psi(\mathrm{x}), \qquad \mathsf{H}_\star^{(1)} := \left[\frac{-\hbar^2}{2m}D^a \star D_a + V\right]\star, \qquad D_a = \partial_a + \mathrm{i}qA_a, \tag{2}$$

one obtains a pseudodifferential equation and therefore introduces a (very special) amount of non-locality in the interactions. A very interesting situation is when $\overline{\mathcal{F}}$ is related to the symmetries of X. If for simplicity $X = \mathbb{R}^m$, $G = ISO(m)$ is the isometry group of X, $\mathbf{g}$ is its Lie algebra, $U\mathbf{g}$ is its universal enveloping algebra (UEA), and we choose $\overline{\mathcal{F}}$ as the inverse of a unitary twist $\mathcal{F} = \mathbf{1}\otimes\mathbf{1}+\lambda\mathcal{F}^\alpha\otimes\mathcal{F}_\alpha+\cdots \in (U\mathbf{g}\otimes U\mathbf{g})[[\lambda]]$ [21] (see also [17, 57]), then the action of $U\mathbf{g}$ on a $\star$-product $f \star h$ obeys (section 3) the deformed Leibniz rule corresponding to the coproduct of the triangular noncocommutative Hopf algebra $\widehat{U\mathbf{g}}$ obtained by twisting $U\mathbf{g}$ with $\mathcal{F}$ (section 2). In a similar way, inverting the λ-power expansion by defining the $\star$-product in all $U\mathbf{g}$-module algebras,

$$f \cdot h = f \star h + \lambda(\mathcal{F}^\alpha \triangleright f) \star (\mathcal{F}_\alpha \triangleright h) + \cdots,$$

we can also express all other commutative notions (wavefunctions ψ, differential operators (Hamiltonian, etc) and integration (section 3.3), $U\mathbf{g}$-covariant a^i, a_i^+ (section 3.4), the wave-mechanical description of n bosons/fermions (section 4.1), etc) purely in terms of their $\star$-analogues and thus translate (sections 4.2 and 4.3) non-relativistic second quantization on X into a 'noncommutative language', which we finally express by the use of 'hatted' objects $\hat{x}^a$, $\hat{\psi}$, etc only; in this final translation, summarized in formulae (43), (50) and (107), all formal λ-power series arising from $\mathcal{F}$ either disappear or are expressed through the triangular structure $\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}$, which has much better representation properties than those of $\mathcal{F}$. In a sense, the philosophy is like the one by Wess and coworkers in formulating [3, 4] noncommutative diffeomorphisms and related notions (metric, connections, tensors, etc).

In a minimalistic view one can see the replacement of all $\cdot$ by $\star$'s in (2) just as a way to introduce non-locality in the interactions, and the use of noncommutative coordinates just as a help to solve this equation, but keep considering spacetime *commutative*, in the sense of describing the measurement processes of the space coordinates of an event still by the (commuting) multiplication operators $\mathrm{x}^a\cdot$. In a more open-minded view, one can reinterpret the results as the construction of a noncommutative space(time) and on it of a 'closed', $\widehat{U\mathbf{g}}$-covariant candidate framework for QFT consistent with the basic principles of quantum mechanics and Bose–Fermi statistics, which is the main purpose of this paper. Its observational consequences (e.g. of adopting noncommutative coordinates $\hat{x}^a$ for describing the measurement processes of the space coordinates of an event, or the meaning of twisted spacetime symmetries) deserve separate investigations.

We anticipate the key features of our deformed non-relativistic QFT framework on $X = \mathbb{R}^3$ as follows. A basic property of φ, φ^* of (1) is their basis independence, i.e. invariance under the group $U(\infty)$ of unitary transformations of $\mathcal{H}$; φ_i, a_i^+ transform according to the same representation ρ; φ_i^*, a^i according to the contragredient $\rho^\vee$. The group $G = ISO(3)$ of (active) space-symmetry transformations (combined translations and rotations of the system) is a subgroup of $U(\infty)$ and of the Galilei covariance group of the theory, G'. Deforming the setting through a twist $\mathcal{F} \in (U\mathbf{g} \otimes U\mathbf{g})[[\lambda]]$ (section 4) must leave the field invariant under $\widehat{U\mathbf{g}}$ (and $\widehat{Uu(\infty)}$). Consequently, deformed a_i^+, a^i no longer commute with deformed functions, but the deformed field does; moreover, the commutation relations of the deformed field appear as the undeformed (section 4.2). The same occurs with Heisenberg fields (defined as usual with a commutative time), and the deformed non-relativistic theory is $\widehat{U\mathbf{g}'}$ covariant (section 4.3).

In section 6 we extend the second quantization procedure to construct relativistic free fields on a deformed Minkowski spacetime covariant under the associated deformed Poincaré symmetry. We then concentrate the attention on the simplest examples, Moyal–Minkowski spaces and the corresponding twisted Poincaré Hopf algebra $\widehat{U\mathcal{P}}$ [15, 47, 59]. It has been debated in [1, 2, 5, 9, 14, 16, 44, 51, 58, 62] on how to implement the $\widehat{U\mathcal{P}}$ covariance even for free fields, three main issues being whether one should also deform the commutation relations (a) among coordinates of different spacetime points, (b) among creation and annihilation operators, (c) of the latter with spacetime coordinates. Our procedure leads to a peculiar combination of (a)–(c) (respectively (43), (71) and (A.8) in the general deformed Minkowski case, (53) and (117) in the Moyal–Minkowski one), but again functions $\star$-commute with quantum fields. It is encouraging that the same combination arises (equation (46) of [32]) also from consistency with the Wightman axioms for relativistic QFT, see also [30][3]. It was also found quite disappointingly in [32] that the n-point functions of a scalar theory, both free and self-interacting, depend on the differences of coordinates at independent spacetime points as in the undeformed theory, i.e. the effect of $\star$-products disappears . This is due to the translation invariance of the interaction $\int \varphi^{\star n}$ considered in [32] and should change in the presence of gauge interactions as in sections 4 and 5.

Our framework is consistent with Bose–Fermi statistics, in contrast with claims often made in the literature [9] and despite changes appearing at two levels. At the level of the deformed wave-mechanical description (sections 4.1 and 6), the realization of the permutation group S_n on noncommutative n-particle wavefunctions differs from the usual one by a unitary transformation $\wedge^n$ related to the twist; as $\wedge^n$ is not completely symmetric, the noncommutative wavefunctions of n bosons/fermions are not completely (anti)symmetric, but are so up to $\wedge^n$, and hence still singlets under this realization of S_n. Such a mechanism, which actually makes Bose–Fermi statistics also compatible with transformations under *quasitriangular* Hopf algebras, was already proposed in [31] on the abstract Hilbert space rather than on its realization on the space of square-integrable wavefunctions, as done here. At the level (section 3.4) of the Heisenberg/Clifford algebra, the Fock-type representation of the deformed algebra is on the *ordinary* (i.e. undeformed) Fock space, which describes the states of a system of bosons/fermions; the deformed a^i, a_i^+ act on the Fock space as 'dressed' operators (i.e. composite in the undeformed ones), but do not map the Fock space out of itself.

Whether our framework is implementable beyond the level of formal λ-power series can be studied case by case using 'noncommutative mathematics' only. On Moyal deformations of space(time)—probably the simplest and best-known noncommutative spaces—this seems to be the case because the $\star$-products admit (section 3.3.1) non-perturbative (in λ) definitions in terms of Fourier transforms.

Sections 2 and 3 give mathematical preliminaries. Section 2 gives a basic introduction to twisting of Hopf algebras H, in particular UEAs, and the associated notation. Section 3 gives a systematic presentation of known and new results regarding the twisting of H-module algebras and its applications; one can focus on the main results in a first reading and come back to

[3] We put aside the other combination proposed in equation (44) of [32] for the reasons presented in [30].

the details when they are referred to in the following sections. As said, sections 4 and 6 treat second quantization respectively of non-relativistic interacting and relativistic non-interacting fields. Section 5 is a detour from field theory to quantum mechanics on the Moyal deformation of $\mathbb{R}^3$. It first shows how the (anti)symmetry of two-particle wavefunctions is translated in terms of noncommutative coordinates; then it illustrates how the occurrence of the $\star$ in (2) modifies the one-particle Schrödinger equation in two simple models (a constant magnetic field and a plane-wave electromagnetic field) and how the use of noncommutative coordinates can help to solve it (in the first model).

We stick to vector spaces V and algebras $\mathcal{A}$ over $\mathbb{C}$. In what follows $V(\mathcal{A})$ stands for the vector space underlying $\mathcal{A}$. $V[[\lambda]]$ and $\mathcal{A}[[\lambda]]$ respectively stand for the topological vector space and algebra (over $\mathbb{C}[[\lambda]]$) of power series in λ with coefficients in $V, \mathcal{A}$; their tensor products are meant as completed in the λ-adic topology.

2. Twisting Hopf algebras *H*

Consider a cocommutative Hopf $*$-algebra $(H, *, \Delta, \varepsilon, S)$; $H, *, \Delta, \varepsilon, S$ stand for the algebra, $*$-structure, coproduct, counit and antipode respectively. For readers who are not so familiar with Hopf algebras, we recall that ε gives the trivial representation, Δ and S are the abstract operations by which one constructs the tensor product of any two representations and the contragredient of any representation, respectively; S is uniquely determined by Δ. We are especially interested in the UEA examples $H = U\mathbf{g}$, with the $*$-structure determined by a real form of the Lie algebra $\mathbf{g}$. Then

$$\begin{aligned} &\varepsilon(\mathbf{1}) = 1, && \Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, && S(\mathbf{1}) = \mathbf{1}, \\ &\varepsilon(g) = 0, && \Delta(g) = g\otimes\mathbf{1}+\mathbf{1}\otimes g, && S(g) = -g, \qquad \text{if } g\in\mathbf{g}; \end{aligned}$$

ε, Δ are extended to all of $H = U\mathbf{g}$ as $*$-algebra maps and S is extended as a $*$-antialgebra map:

$$\begin{aligned} &\varepsilon : H \mapsto \mathbb{C}, && \varepsilon(ab) = \varepsilon(a)\varepsilon(b), && \varepsilon(a^*) = [\varepsilon(a)]^*, \\ &\Delta : H \mapsto H\otimes H, && \Delta(ab) = \Delta(a)\Delta(b), && \Delta(a^*) = [\Delta(a)]^{*\otimes *}, \\ &S : H \mapsto H, && S(ab) = S(b)S(a), && S\{[S(a^*)]^*\} = a. \end{aligned} \tag{3}$$

The extensions of Δ, S are unambiguous, as $\Delta([g, g']) = [\Delta(g), \Delta(g')]$, $S([g, g']) = [S(g'), S(g)]$ if $g, g' \in \mathbf{g}$. We shall often abbreviate $(H, *, \Delta, \varepsilon, S)$ just as H. Clearly $(H[[\lambda]], *, \Delta, \varepsilon, S)$ is a (topological) cocommutative Hopf $*$-algebra if we extend the product, Δ, ε, S $\mathbb{C}[[\lambda]]$-linearly and $*$ $\mathbb{C}[[\lambda]]$-antilinearly.

Consider a *twist* [21] (see also [17, 57]), i.e. an element $\mathcal{F} \in (H\otimes H)[[\lambda]]$ fulfilling[4]

$$\mathcal{F} \equiv \sum_I \mathcal{F}_I^{(1)}\otimes\mathcal{F}_I^{(2)} = \mathbf{1}\otimes\mathbf{1} + O(\lambda), \qquad (\epsilon\otimes\mathrm{id})\mathcal{F} = (\mathrm{id}\otimes\epsilon)\mathcal{F} = \mathbf{1}, \tag{4}$$

$$(\mathcal{F}\otimes\mathbf{1})[(\Delta\otimes\mathrm{id})(\mathcal{F})] = (\mathbf{1}\otimes\mathcal{F})[(\mathrm{id}\otimes\Delta)(\mathcal{F})] =: \mathcal{F}^3 \quad \text{(cocycle condition)}. \tag{5}$$

Let $H_s \subseteq H$ be the smallest Hopf $*$-subalgebra such that $\mathcal{F} \in (H_s\otimes H_s)[[\lambda]]$. Let also

$$\beta := \sum_I \mathcal{F}_I^{(1)} S\big(\mathcal{F}_I^{(2)}\big) \in H_s[[\lambda]]. \tag{6}$$

[4] By definition $\mathcal{F} = \sum_{k=0}^{\infty} f_k\lambda^k$ with $f_k \in H\otimes H$; $f_0 = \mathbf{1}\otimes\mathbf{1}$ by the second equality in $(4)_1$. Fixing a basis $\{h_\mu\}$ of H, f_k can be decomposed as a finite combination $f_k = \sum_{\mu,\nu} f_k^{\mu\nu} h_\mu\otimes h_\nu$, with $f_k^{\mu\nu} \in \mathbb{C}$. The generic term $\mathcal{F}_I^{(1)}\otimes\mathcal{F}_I^{(2)}$ in the decomposition of $\mathcal{F}$ is $\lambda^k f_k^{\mu\nu} h_\mu\otimes h_\nu$, and $\sum_I$ means $\sum_{k,\mu,\nu}$.

For our purposes $\mathcal{F}$ is *unitary* ($\mathcal{F}^{*\otimes *} = \mathcal{F}^{-1}$), whence also β is ($\beta^* = \beta^{-1}$); without loss of generality λ can be assumed real. Let $\hat{H} := H[[\lambda]]$ and for any $g \in \hat{H}$[5]

$$g^{\hat{*}} := g^*, \qquad \hat{\Delta}(g) := \mathcal{F}\Delta(g)\mathcal{F}^{-1}, \qquad \hat{\varepsilon}(g) := \varepsilon(g), \qquad \hat{S}(g) := \beta S(g)\beta^{-1}; \tag{8}$$

one finds that the analogues of conditions (3) are satisfied, and therefore $(\hat{H}, \hat{*}, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$ is a Hopf $*$-algebra (which we shall often abbreviate as $\hat{H}$) with the triangular structure $\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}$, deformation of the initial one. $(\hat{H}_s, \hat{*}, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$, with $\hat{H}_s := H_s[[\lambda]]$, is a Hopf $*$-subalgebra. Drinfel'd has shown [21] that any triangular deformation of the initial Hopf algebra can be obtained in this way (up to isomorphisms). Given $\hat{\Delta}, \Delta$, the twist $\mathcal{F}$ is determined up to the multiplication $\mathcal{F} \to \mathcal{F}' = \mathcal{F}T$ by a unitary and **g**-invariant (i.e. commuting with $\Delta(\mathbf{g})$) element $T \in (H \otimes H)[[\lambda]]$ fulfilling (4), (5)[6]. $\mathcal{R}$ is independent of T if the latter is symmetric. Suitable additional conditions may restrict the choice or uniquely determine T and hence the twist.

Equations (5) and (8) imply the generalized intertwining relation $\hat{\Delta}^{(n)}(g) = \mathcal{F}^n\Delta^{(n)}(g)(\mathcal{F}^n)^{-1}$ for the iterated coproduct. By definition

$$\mathcal{F}^n \in (\hat{H}_s)^{\otimes n}, \qquad \Delta^{(n)} : H \mapsto H^{\otimes n}, \qquad \hat{\Delta}^{(n)} : \hat{H} \mapsto \hat{H}^{\otimes n}$$

reduce to $\mathcal{F}, \Delta, \hat{\Delta}$ for $n = 2$, whereas for $n > 2$ they can be defined recursively as

$$\begin{aligned} &\mathcal{F}^{m+1} = (\mathbf{1}^{\otimes(m-1)} \otimes \mathcal{F})[(\mathrm{id}^{\otimes(m-1)} \otimes \Delta)\mathcal{F}^m], \\ &\Delta^{(m+1)} = (\mathrm{id}^{\otimes(m-1)} \otimes \Delta) \circ \Delta^{(m)}, \qquad \hat{\Delta}^{(m+1)} = (\mathrm{id}^{\otimes^{m-1}} \otimes \hat{\Delta}) \circ \hat{\Delta}^{(m)}. \end{aligned} \tag{9}$$

The results for $\mathcal{F}^n, \Delta^{(n)}, \hat{\Delta}^{(n)}$ are the same if at any step m we respectively apply $\mathcal{F}(\cdot\otimes\cdot)\Delta, \Delta, \hat{\Delta}$ not to the last but to a different tensor factor; for $n = 3$ this means that (5) and the equalities $\Delta^{(3)} = (\Delta \otimes \mathrm{id}) \circ \Delta$, $\hat{\Delta}^{(3)} = (\hat{\Delta} \otimes \mathrm{id}) \circ \hat{\Delta}$ hold (coassociativity of $\Delta, \hat{\Delta}$). For any $g \in \hat{H} = H[[\lambda]]$ we shall use the following Sweedler notations for the decompositions of $\Delta^{(n)}(g), \hat{\Delta}^{(n)}(g)$ into $\hat{H}^{\otimes n}$[7]:

$$\Delta^{(n)}(g) = \sum_I g^I_{(1)} \otimes g^I_{(2)} \otimes \cdots \otimes g^I_{(n)}, \qquad \hat{\Delta}^{(n)}(g) = \sum_I g^I_{(\hat{1})} \otimes g^I_{(\hat{2})} \otimes \cdots \otimes g^I_{(\hat{n})}.$$

3. Twisting *H*-modules and *H*-module algebras

We recall that, given a Hopf $*$-algebra H over $\mathbb{C}$, a left H-module $(\mathcal{M}, \triangleright)$ is a vector space $\mathcal{M}$ over $\mathbb{C}$ equipped with a left action, i.e. a $\mathbb{C}$-bilinear map $(g, a) \in H \times \mathcal{M} \mapsto g \triangleright a \in \mathcal{M}$ such that $(10)_1$ holds. Also equipped with an antilinear involution $*$ fulfilling $(10)_2$ $(\mathcal{M}, \triangleright, *)$ is a left H-$*$-module. Finally, a left H-module ($*$-)algebra is a $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a left H-($*$-)module structure $\big(V(\mathcal{A}), \triangleright, *\big)$ such that $(10)_3$ holds:

$$(gg') \triangleright a = g \triangleright (g' \triangleright a), \qquad (g \triangleright a)^* = [S(g)]^* \triangleright a^*, \qquad g \triangleright (ab) = \sum_I \big(g^I_{(1)} \triangleright a\big)\big(g^I_{(2)} \triangleright b\big). \tag{10}$$

[5] In (8) one could replace β^{-1} by $S(\beta)$, as $S(\beta)\beta \in \mathrm{Centre}(H)[[\lambda]]$. In terms of the decomposition $\overline{\mathcal{F}} \equiv \mathcal{F}^{-1} = \sum_I \overline{\mathcal{F}}^{(1)}_I \otimes \overline{\mathcal{F}}^{(2)}_I$ one can show that

$$\beta^{-1} = \sum_I S\big(\overline{\mathcal{F}}^{(1)}_I\big)\overline{\mathcal{F}}^{(2)}_I, \qquad S\big(\beta^{-1}\big) = \sum_I S\big(\overline{\mathcal{F}}^{(2)}_I\big)\overline{\mathcal{F}}^{(1)}_I. \tag{7}$$

[6] On the other hand, if we know a non-unitary twist for the Hopf $*$-algebra $\hat{H}$, a transformation of this kind allows us to construct [22, 41] a unitary 1: it suffices to choose $T = (\mathcal{F}^{*\otimes *}\mathcal{F})^{-1/2}$.

[7] This is the analogue of $(4)_1$, i.e. $\sum_I$ means $\sum_{k,\mu,\nu,\ldots}$, etc.

If $g \in \mathbf{g}$, formula (10)$_3$ becomes the Leibniz rule. Given a Hopf ($*$-)algebra $\hat{H}$ the analogous objects (over $\mathbb{C}[[\lambda]]$) and maps are defined putting a $\hat{}$ over the previous symbols. Hereby, (10) is replaced by

$$(gg')\hat{\triangleright}\hat{a} = g\hat{\triangleright}(g'\hat{\triangleright}\hat{a}), \qquad (g\hat{\triangleright}\hat{a})^{\hat{*}} = [\hat{S}(g)]^{\hat{*}}\hat{\triangleright}\hat{a}^{\hat{*}}, \qquad g\hat{\triangleright}(\hat{a}\hat{b}) = \sum_I \big(g^I_{(\hat{1})}\hat{\triangleright}\hat{a}\big)\big(g^I_{(\hat{2})}\hat{\triangleright}\hat{b}\big). \tag{11}$$

(11)$_3$ gives the deformation of the Leibniz rule. Extending the action $\triangleright$ $\mathbb{C}[[\lambda]]$-bilinearly one can trivially extend any H-module $(\mathcal{M}, \triangleright)$ to an $\hat{H}$-module $(\mathcal{M}[[\lambda]], \triangleright)$. If $(\mathcal{M}, \triangleright, *)$ is an H-$*$-module and $\mathcal{F}$ is unitary, then $(\mathcal{M}[[\lambda]], \triangleright, *_\star)$ with

$$a^{*_\star} := S(\beta) \triangleright a^* \tag{12}$$

is an $\hat{H}$-$*$-module[8]. Given an H-module ($*$-)algebra $\mathcal{A}$ and choosing $\mathcal{M} = V(\mathcal{A})$, the twist also provides a systematic way to make $V(\mathcal{A})[[\lambda]]$ an $\hat{H}$-module ($*$-)algebra $\mathcal{A}_\star$ by endowing it with a new product, the $\star$-product, defined by

$$a \star b := \sum_I \big(\overline{\mathcal{F}}_I^{(1)} \triangleright a\big)\big(\overline{\mathcal{F}}_I^{(2)} \triangleright b\big). \tag{13}$$

The $\mathbb{C}[[\lambda]]$-bilinearity of $\star$ is manifest and the associativity follows from (5), whereas

$$g \triangleright (a \star b) \overset{(10)_3}{=} \sum_{I,I'} \big(g^I_{(1)}\overline{\mathcal{F}}_{I'}^{(1')} \triangleright a\big)\big(g^I_{(2)}\overline{\mathcal{F}}_{I'}^{(2')} \triangleright b\big) \overset{(8)}{=} \sum_{I,I'} \big(\overline{\mathcal{F}}_{I'}^{(1')} g^I_{(\hat{1})} \triangleright a\big)\big(\overline{\mathcal{F}}_{I'}^{(2')} g^I_{(\hat{2})} \triangleright b\big)$$
$$= \sum_I \big(g^I_{(\hat{1})} \triangleright a\big) \star \big(g^I_{(\hat{2})} \triangleright b\big)$$

proves property (11)$_3$. In the appendix we prove the compatibility with $*_\star$,

$$(a \star b)^{*_\star} = b^{*_\star} \star a^{*_\star}. \tag{14}$$

By (4), the $\star$-product coincides with the original one if a or b is H_s-invariant:

$$g \triangleright a = \epsilon(g)a \quad \text{or} \quad g \triangleright b = \epsilon(g)b \quad \forall g \in H_s \quad \Rightarrow \quad a \star b = ab. \tag{15}$$

In the literature $\star$-products are mostly introduced to deform Abelian algebras (e.g. the algebra of functions on a manifold) into non-Abelian ones. We stress that the above construction works also if $\mathcal{A}$ is non-Abelian, e.g. if $\mathcal{A} = H$ [4, 21].

Given two H-modules $(\mathcal{M}, \triangleright)$, $(\mathcal{N}, \triangleright)$, the tensor product $(\mathcal{M} \otimes \mathcal{N}, \triangleright)$ also is an H-module, if we define $g \triangleright (a \otimes b) := \sum_I \big(g^I_{(1)} \triangleright a\big) \otimes \big(g^I_{(2)} \triangleright b\big)$. As stated above, this is extended to an $\hat{H}$-($*$-)module $(\mathcal{M} \otimes \mathcal{N}[[\lambda]], \triangleright)$. Introducing the '$\star$-tensor product' [3, 4]

$$(a \otimes_\star b) := \mathcal{F}^{-1} \triangleright^{\otimes 2} (a \otimes b) \equiv \sum_I \big(\overline{\mathcal{F}}_I^{(1)} \triangleright a\big) \otimes \big(\overline{\mathcal{F}}_I^{(2)} \triangleright b\big) \tag{16}$$

(an invertible endomorphism, i.e. a change of bais, of $\mathcal{M} \otimes \mathcal{N}[[\lambda]]$), we find

$$g \triangleright (a \otimes_\star b) = \sum_I \big(g^I_{(\hat{1})} \triangleright a\big) \otimes_\star \big(g^I_{(\hat{2})} \triangleright b\big). \tag{17}$$

Note that this has the same form as the law $g\hat{\triangleright}(a \otimes b) = \sum_I \big(g^I_{(\hat{1})}\hat{\triangleright}a\big) \otimes \big(g^I_{(\hat{2})}\hat{\triangleright}b\big)$ used to build an $\hat{H}$-module $(\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}, \hat{\triangleright})$ out of two $(\hat{\mathcal{M}}, \hat{\triangleright})$, $(\hat{\mathcal{N}}, \hat{\triangleright})$. Given two H-module ($*$-)algebras $\mathcal{A}, \mathcal{B}$, this applies in particular to $\mathcal{M} = V(\mathcal{A})$, $\mathcal{N} = V(\mathcal{B})$. The tensor ($*$-)algebra $\mathcal{A} \otimes \mathcal{B}$ (whose product is defined by $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$) is also an H-module ($*$-)algebra under the action $\triangleright$. By introducing the $\star$-product (13) $\mathcal{A} \otimes \mathcal{B}$ is deformed into an $\hat{H}$-module

[8] See the appendix. It is not possible to keep $*_\star = *$ as in [4], section 8, since that was based on the condition $\mathcal{F}^{*\otimes *} = (S \otimes S)(\mathcal{F}_{21})$ rather than $\mathcal{F}^{*\otimes *} = \mathcal{F}^{-1}$.

($*$-)algebra $(\mathcal{A}\otimes\mathcal{B})_\star$. To clarify the relation with the products (and $*$-structures) in $\mathcal{A}_\star,\mathcal{B}_\star$ we compute (see the appendix)

$$\begin{aligned}(a\otimes_\star b)\star(a'\otimes_\star b') &= \textstyle\sum_I a\star(\mathcal{R}_I^{(2)}\triangleright a')\otimes_\star(\mathcal{R}_I^{(1)}\triangleright b)\star b',\\ (a\otimes_\star b)^{*_\star} &= \textstyle\sum_I\left(\mathcal{R}_I^{(2)}\triangleright a\right)\otimes_\star\left(\mathcal{R}_I^{(1)}\triangleright b\right),\end{aligned}\tag{18}$$

where $\sum_I\mathcal{R}_I^{(1)}\otimes\mathcal{R}_I^{(2)}$ is the decomposition of $\mathcal{R}$ into $\hat{H}\otimes\hat{H}$. Note that if $\mathcal{A},\mathcal{B}$ are unital[9], then $a\otimes_\star b=a_1\star b_2$ and $(a\otimes_\star b)^{*_\star}=b_2^{*_\star}\star a_1^{*_\star}$, with the short-hand notation $a_1:=a\otimes\mathbf{1}_\mathcal{B}$, $b_2:=\mathbf{1}_\mathcal{A}\otimes b$. From (18) we recognize that $(\mathcal{A}\otimes\mathcal{B})_\star$ is isomorphic to the *braided tensor product (algebra)* [17, 45] of $\mathcal{A}_\star$ with $\mathcal{B}_\star$; here the braiding is involutive and therefore spurious, as $\mathcal{R}\mathcal{R}_{21}=\mathbf{1}\otimes\mathbf{1}$ (triangularity of $\mathcal{R}$). So $(\mathcal{A}\otimes\mathcal{B})_\star$ encodes both the usual $\star$-product within $\mathcal{A},\mathcal{B}$ and the $\star$-tensor product (or the braided tensor product) between the two. By (5) the $\star$-tensor product is associative, and the previous results also hold for iterated $\star$-tensor products.

3.1. Module algebras defined by generators and relations

Here we show that if $\mathcal{A}$ is defined by generators and relations, then also $\mathcal{A}_\star$ is, with the same generators and Poincaré–Birkhoff–Witt series, and with related relations; and similarly for tensor products.

Given a generic H-($*$-)module $\mathcal{M}$ and fixing a (for simplicity discrete) basis $\{a_i\}_{i\in\mathcal{I}}$ of $\mathcal{M}$, consider the free algebra $\mathcal{A}^f$ (over $\mathbb{C}$) with $\{a_i\}_{i\in\mathcal{I}}$ as a set of generators. $\mathcal{A}^f$ is also automatically an H-module ($*$-)algebra under the action

$$g\triangleright(a_{i_1}a_{i_2}\cdots a_{i_k})=\sum_I\left(g^I_{(1)}\triangleright a_{i_1}\right)\left(g^I_{(2)}\triangleright a_{i_2}\right)\cdots\left(g^I_{(k)}\triangleright a_{i_k}\right)$$

and has the spaces $\mathcal{M}^k$ of homogeneous polynomials of degree k as H-($*$-)submodules. Endowing $V(\mathcal{A}^f)[[\lambda]]$ with the $\star$-product (13) one deforms $\mathcal{A}^f$ into an $\hat{H}$-module ($*$-)algebra $\mathcal{A}^f_\star$ having the spaces $\mathcal{M}^k_\star$ of homogeneous $\star$-polynomials of degree k as $\hat{H}$-($*$-)submodules. Inverting the definition of the generic $\star$-monomial of degree k one can express the generic monomial of degree k as a homogeneous $\star$-polynomial of degree k:

$$a_{i_1}a_{i_2}\cdots a_{i_k}=F^{k\,j_1j_2\cdots j_k}_{\;\;i_1i_2\cdots i_k}a_{j_1}\star a_{j_2}\star\cdots\star a_{j_k}\tag{19}$$

(here F^k is the $\mathbb{C}[[\lambda]]$-valued matrix defined by $F^k=\tau^{\otimes k}(\mathcal{F}^k)$, τ being the representation defined by $g\triangleright a_i=\tau_i^j(g)a_j$). Formula (19) is an identity in $V(\mathcal{A}^f)[[\lambda]]=V\left(\mathcal{A}^f_\star\right)$ which can be used to establish a one-to-one correspondence $\wedge: f\in\mathcal{A}^f[[\lambda]]\mapsto\wedge(f)\equiv\hat{f}\in\mathcal{A}^f_\star$ by the requirement that the latter reduces to the identity on $V(\mathcal{A}^f)[[\lambda]]=V\left(\mathcal{A}^f_\star\right)$: the polynomials $f,\hat{f}$

$$f=\sum_{k=0}^m f_k^{j_1\cdots j_k}a_{j_1}\cdots a_{j_k},\qquad \hat{f}=\sum_{k=0}^n\hat{f}_k^{j_1\cdots j_k}a_{j_1}\star\cdots\star a_{j_k}\qquad f_k^{j_1\cdots j_k},\hat{f}_k^{j_1\cdots j_k}\in\mathbb{C}[[\lambda]],$$

(sum over repeated indices j_h is understood) are such that the equality

$$f=\hat{f}\tag{20}$$

holds in $V(\mathcal{A}^f)[[\lambda]]=V(\mathcal{A}^f_\star)$ (then clearly their degrees m,n must coincide); $\wedge$ is $\mathbb{C}[[\lambda]]$-linear by construction. If e.g. f is the monomial on the lhs(19), then $\hat{f}$ is the rhs(19). Using (5) it is easy to show that $\wedge,\wedge^{-1}$ fulfil

$$\begin{aligned}\wedge(ff') &= \textstyle\sum_I\wedge\left[\mathcal{F}_I^{(1)}\triangleright f\right]\star\wedge\left[\mathcal{F}_I^{(2)}\triangleright f'\right],\\ \wedge^{-1}(\hat{f}\star\hat{f}') &= \textstyle\sum_I\left[\overline{\mathcal{F}}_I^{(1)}\triangleright\wedge^{-1}(\hat{f})\right]\left[\overline{\mathcal{F}}_I^{(2)}\triangleright\wedge^{-1}(\hat{f}')\right].\end{aligned}\tag{21}$$

[9] This is no real restriction: a non-unital algebra can be always extended unitally.

Assume that $\mathcal{A} = \mathcal{A}^f/\mathcal{I}$, where $\mathcal{I}$ is an H-invariant $(*\text{-})$ideal generated by a set of polynomial relations[10]

$$f^J = 0, \qquad J \in \mathcal{J}. \tag{22}$$

Imposing (22), in particular those of degree k, will make $\mathcal{M}^k$ an H-$(*\text{-})$submodule $\mathcal{M}'^k$ consisting of (no longer necessarily homogeneous) polynomials of degree k. The $\star$-polynomial relations

$$\hat{f}^J = 0, \qquad J \in \mathcal{J}, \tag{23}$$

will generate an $\hat{H}$-invariant ideal $\mathcal{I}_\star$; therefore, $\mathcal{A}_\star := \mathcal{A}^f_\star\big/\mathcal{I}_\star$ is an $\hat{H}$-module $(*\text{-})$algebra with generators a_i and relations (23). Imposing the latter $\mathcal{M}^k_\star$ become $\hat{H}$-$(*\text{-})$submodules $\mathcal{M}'^k_\star$ consisting of corresponding suitable $\star$-polynomials of degree k. By construction the Poincaré–Birkhoff–Witt series of $\mathcal{A}$, $\mathcal{A}_\star$ coincide. Given a polynomial $f \in \mathcal{A}[[\lambda]]$, we shall denote still by $\hat{f} \in \mathcal{A}_\star$ the polynomial such that equality (20) holds now in $V(\mathcal{A}_\star) = V(\mathcal{A})[[\lambda]]$, and by $\wedge : f \in \mathcal{A}[[\lambda]] \mapsto \hat{f} \in \mathcal{A}_\star$ the corresponding linear map; for the latter (21) remains valid. Summing up, *by* $\wedge, \wedge^{-1}$ *one respectively expresses polynomials in the* a_i as $\star$-polynomials in the a_i, and vice versa.

Denoting by $\{a_i\}_{i\in\mathcal{I}}$, $\{b'_i\}_{i'\in\mathcal{I}'}$ sets of generators of unital $\mathcal{A}$, $\mathcal{B}$, a set of generators of both $\mathcal{A}\otimes\mathcal{B}$ and $(\mathcal{A}\otimes\mathcal{B})_\star$ will consist of $\{a_{i1}, b_{i'2}\}_{i\in\mathcal{I},i'\in\mathcal{I}'}$. As generators of $\mathcal{A}\otimes\mathcal{B}$ (resp. $(\mathcal{A}\otimes\mathcal{B})_\star$) the a_{i1} fulfil (22) (resp. (23)), the $b_{i'2}$ the analogous relations for $\mathcal{B}$, and

$$a_{i1}b_{i'2} = b_{i'2}a_{i1} \qquad \left(\text{resp. } a_{i1} \star b_{i'2} = \sum_I \left(\mathcal{R}^{(2)}_I \triangleright b_{i'2}\right) \star \left(\mathcal{R}^{(1)}_I \triangleright a_{i1}\right)\right). \tag{24}$$

The a_{i1} generate an H- (resp. $\hat{H}$-) module (*-)subalgebra, which we call $\mathcal{A}_1$ (resp. $\mathcal{A}_{1\star}$). As an H-(resp. $\hat{H}$-) module (*-)algebra, this is isomorphic to $\mathcal{A}$ (resp. $\mathcal{A}_\star$) and similarly for $\mathcal{B}_2$ (resp. $\mathcal{B}_{2\star}$).

Finally note that as in (23), $(24)_2$ the original product of $\mathcal{A}$ does not appear any longer, one can introduce $\mathcal{A}_\star$, $\mathcal{B}_\star$, $(\mathcal{A}\otimes\mathcal{B})_\star$ just in terms of these generators and relations. For convenience we shall denote them as $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$, $\widehat{\mathcal{A}\otimes\mathcal{B}}$, the generators as $\hat{a}_i$, $\hat{b}_{i'}$, $\hat{a}_{i1}$, $\hat{b}_{i'2}$, $*_\star$ as $\hat{*}$ and omit the $\star$-product symbols; for example, $\widehat{\mathcal{A}}$ can be abstractly defined as the algebra generated by $\hat{a}_i$ fulfilling the relations obtained from (23) by replacing $a_i \to \hat{a}_i$ and dropping $\star$,

$$\hat{f}^J(\hat{a}_1, \hat{a}_2, \cdots) = 0, \qquad J \in \mathcal{J}, \tag{25}$$

with $*$-structure, if any, defined by $\hat{a}_i^{\hat{*}} = \tau^h_j[S(\beta)]K^j_i a_h$, where $K^j_i \in \mathbb{C}$ are defined by $a^*_i = K^j_i a_j$ ($K^j_i = \delta^j_i$ if a_i are real). On $\widehat{\mathcal{A}}$ the action $\triangleright$ fulfils $(11)_3$.

3.2. Deforming maps

Here we describe how an $\hat{H}$-module $(*)$-algebra $\mathcal{A}_\star$ can be realized within $\mathcal{A}[[\lambda]]$ (i.e. how its elements can be realized as power series in λ with coefficients in $\mathcal{A}$) through a so-called *deforming map*, provided the left H-module $(*)$-algebra $\mathcal{A}$ admits a $(*)$-algebra map

[10] This assumption is satisfied in most cases of interest. For instance, if $\mathcal{A}$ has a unit then the latter must be one of the a_i, say $a_0 = \mathbf{1}$, must span a one-dimensional submodule, and among these relations there must be $\mathbf{1}a_i - a_i = a_i\mathbf{1} - a_i = 0$ for all i. If $\mathcal{A}$ is a UEA of a Lie algebra, the remaining relations are of the form $a_ia_j - a_ja_i = c^k_{ij}a_k$. If instead $\mathcal{A}$ is Abelian among these relations, there must be $a_ia_j - a_ja_i = 0$, for all i, j. Imposing further polynomial relations one can obtain the algebra of polynomial functions on an algebraic manifold. The algebra of differential operators on the latter is obtained adding, as generators, a basis of vector fields (first-order differential operators) and as additional relations the Lie algebra ones which are fulfilled by the latter together with the Leibniz-rule-type commutation relations and the functions on the manifold. Similarly, superalgebras can be defined by replacing some of the above commutation relations by anticommutation relations, and so on.

$\sigma : H \mapsto \mathcal{A}$ such that the H-action on $\mathcal{A}$ can be expressed in the (cocommutative) left 'adjoint-like' form

$$g \triangleright a = \sum_I \sigma\big(g^I_{(1)}\big) a \sigma\big(S g^I_{(2)}\big). \tag{26}$$

This will hold of course also for the corresponding linear extensions $\sigma : \hat{H} = H[[\lambda]] \to \mathcal{A}[[\lambda]]$ and $\triangleright : H[[\lambda]] \times \mathcal{A}[[\lambda]] \mapsto \mathcal{A}[[\lambda]]$, and suggest a way to make $\mathcal{A}[[\lambda]]$ an $\hat{H}$-module $*$-algebra by defining the corresponding action $\hat{\triangleright}$ in the (noncocommutative) 'adjoint-like' form

$$g \hat{\triangleright} a := \sum_I \sigma\big(g^I_{(\hat{1})}\big) a \sigma\big(\hat{S} g^I_{(\hat{2})}\big). \tag{27}$$

In case $\mathcal{A} = H$, then $\sigma = \text{id}$, (26) is the adjoint action of H, and the action defined by (27) makes $H[[\lambda]]$ an $\hat{H}$-module $*$-algebra—see the end of this subsection. It is easy to check that the *deforming map* [28] $D^\sigma_\mathcal{F} : a \in V(\mathcal{A})[[\lambda]] \mapsto \check{a} \in V(\mathcal{A})[[\lambda]]$ defined by

$$\check{a} \equiv D^\sigma_\mathcal{F}(a) := \sum_I \sigma\big(\mathcal{F}^{(1)}_I\big) a\, \sigma\big[S\big(\mathcal{F}^{(2)}_I\big)\beta^{-1}\big] \tag{28}$$

has the interesting property of intertwining between $\triangleright, \hat{\triangleright}$:

$$g \hat{\triangleright} \big[D^\sigma_\mathcal{F}(a)\big] = D^\sigma_\mathcal{F}(g \triangleright a). \tag{29}$$

An immediate consequence is that if $\mathcal{M} \subseteq V(\mathcal{A})$ is an H-$*$-submodule, then $D^\sigma_\mathcal{F}(\mathcal{M})$ is an $\hat{H}$-$*$-submodule. An alternative expression for $D^\sigma_\mathcal{F}$ is (cf with [4, 36, 39])

$$\check{a} \equiv D^\sigma_\mathcal{F}(a) = \sum_I \big(\overline{\mathcal{F}}^{(1)}_I \triangleright a\big) \sigma\big(\overline{\mathcal{F}}^{(2)}_I\big). \tag{30}$$

Moreover,

$$\big[D^\sigma_\mathcal{F}(a)\big]^* = D^\sigma_\mathcal{F}[a^{*_\star}] \tag{31}$$

(we emphasize that the undeformed $*$ is on the lhs), implying (with the help of (12))

$$(g \,\hat{\triangleright}\, \check{a})^* = [\hat{S}(g)]^* \,\hat{\triangleright}\, (\check{a})^*.$$

Finally, $D^\sigma_\mathcal{F}$ also satisfies

$$D^\sigma_\mathcal{F}(a \star a') = D^\sigma_\mathcal{F}(a) D^\sigma_\mathcal{F}(a') \tag{32}$$

(equations (30)–(32) are proved in the appendix). Summing up, *the deforming map is an $\hat{H}$-module $*$-algebra isomorphism* $D^\sigma_\mathcal{F} : \mathcal{A}_\star \leftrightarrow \mathcal{A}[[\lambda]]$, or, changing notation, $\widehat{D}^\sigma_\mathcal{F} : \widehat{\mathcal{A}} \leftrightarrow \mathcal{A}[[\lambda]]$.

If $\mathcal{A}^s \subseteq \mathcal{A}$ is an H-module ($*$-)subalgebra, then $\check{\mathcal{A}}^s := D^\sigma_\mathcal{F}(\mathcal{A}^s) \subseteq \mathcal{A}[[\lambda]]$ is an $\hat{H}$-module ($*$-)subalgebra isomorphic to $\mathcal{A}^s_\star$, with $\hat{H}$-module $*$-algebra isomorphism $D^\sigma_\mathcal{F} : \mathcal{A}^s_\star \mapsto \check{\mathcal{A}}^s$. Also note that by (4), (30), $\check{a} = a$ if a is H_s-invariant:

$$g \triangleright a = \epsilon(g) a \quad \forall g \in H_s \quad \Rightarrow \quad D^\sigma_\mathcal{F}(a) = a. \tag{33}$$

From (28) the inverse of $D^\sigma_\mathcal{F}$ is readily seen to be

$$\big(D^\sigma_\mathcal{F}\big)^{-1}(a) := \sum_I \sigma\big(\overline{\mathcal{F}}^{(1)}_I\big) a\, \sigma\big[\beta S\big(\overline{\mathcal{F}}^{(2)}_I\big)\big]. \tag{34}$$

If $\mathcal{A}, \mathcal{B}$ are H-module $*$-algebras admitting $*$-algebra maps $\sigma_\mathcal{A} : H \mapsto \mathcal{A}$, $\sigma_\mathcal{B} : H \mapsto \mathcal{B}$ fulfilling (26), then

$$\sigma_{\mathcal{A}\otimes\mathcal{B}} := (\sigma_\mathcal{A} \otimes \sigma_\mathcal{B}) \circ \Delta, \qquad \hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}} := (\sigma_\mathcal{A} \otimes \sigma_\mathcal{B}) \circ \hat{\Delta}$$

define $*$-algebra maps $\sigma_{\mathcal{A}\otimes\mathcal{B}} : H \mapsto \mathcal{A}\otimes\mathcal{B}$, $\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}} : \hat{H} \mapsto (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$. The former fulfils the analogue of (26), whereas replacing in (27) σ by the latter and a by $c \in (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$,

$$g\hat{\triangleright}c := \sum_I \hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}\big(g^I_{(\hat{1})}\big)\, c\, \hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}\big[\hat{S}\big(g^I_{(\hat{2})}\big)\big]$$

defines the action $\hat{\triangleright}$ making $(\mathcal{A}\otimes\mathcal{B})[[\lambda]]$ into an $\hat{H}$-module $(*\text{-})$algebra; incidentally, note that $g\hat{\triangleright}(a\otimes b) \neq \sum_I \big(g^I_{(\hat{1})}\triangleright a\big)\otimes\big(g^I_{(\hat{2})}\triangleright b\big)$. In the appendix we prove that the corresponding deforming map $D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}} : (\mathcal{A}\otimes\mathcal{B})_\star \mapsto (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$ is defined by

$$D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(c) := \mathcal{F}_\sigma \sum_I \big(\overline{\mathcal{F}}_I^{(1)}\triangleright c\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}_I^{(2)}\big)\mathcal{F}_\sigma{}^{-1}, \qquad \mathcal{F}_\sigma := (\sigma_{\mathcal{A}}\otimes\sigma_{\mathcal{B}})(\mathcal{F}). \tag{35}$$

Using $(\mathrm{id}\otimes\hat{\Delta})(\overline{\mathcal{R}}) = \overline{\mathcal{R}}_{12}\overline{\mathcal{R}}_{13}$ and (27) and setting $\mathcal{R}_\sigma = (\sigma_{\mathcal{A}}\otimes\sigma_{\mathcal{B}})(\mathcal{R})$, we also prove that

$$\check{\alpha}_1 = \check{\alpha}\otimes\mathbf{1}, \qquad \check{b}_2 = \overline{\mathcal{R}}_\sigma(\mathbf{1}\otimes\check{b})\mathcal{R}_\sigma = \sum_I \sigma_{\mathcal{A}}\big(\overline{\mathcal{R}}_I^{(1)}\big)\otimes\overline{\mathcal{R}}_I^{(2)}\hat{\triangleright}\check{b} \tag{36}$$

(here $\overline{\mathcal{R}} := \mathcal{R}^{-1} \equiv \mathcal{R}_{21}$). If $\mathcal{A}$ is generated by a set of H-equivariant $\{a_i\}_{i\in\mathcal{I}}$ fulfilling (22), we find that $\check{a}_i := D_{\mathcal{F}}^{\sigma}(a_i)$, which make up an alternative set of generators of $\mathcal{A}[[\lambda]]$, in fact span an $\hat{H}$-submodule and close the polynomial relations (25), so that they provide an explicit realization of $\widehat{\mathcal{A}} \sim \mathcal{A}_\star$ within $\mathcal{A}[[\lambda]]$. Therefore, $D_{\mathcal{F}}^{\sigma}$ can be seen as a change from a set of H-equivariant to a set of $\hat{H}$-equivariant generators of $\mathcal{A}[[\lambda]]$. Applying $D_{\mathcal{F}}^{\sigma}$ to (20) (with constraints (22), (23)) one finds

$$D_{\mathcal{F}}^{\sigma}[f(a_1, a_2, \cdots)] = \hat{f}(\check{a}_1, \check{a}_2, \cdots)]. \tag{37}$$

Similarly, defining

$$\check{a}_{i1} := D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(a_{i1}) \qquad \check{b}_{i'2} := D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(b_{i'2})$$

we find that $\{\check{a}_{i1}, \check{b}_{i'2}\}_{(i,i')\in\mathcal{I}\times\mathcal{I}'}$ is an alternative set of generators of $(\mathcal{A}\otimes\mathcal{B})[[\lambda]]$. $\{\check{a}_{i1}\}_{i\in\mathcal{I}}$ fulfil (25) and generate an $\hat{H}$-module $(*\text{-})$subalgebra $\check{\mathcal{A}}_1 := D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(\mathcal{A}_1)$, isomorphic to $\mathcal{A}_\star$ as an $\hat{H}$-module $(*\text{-})$algebra. Similarly, $\{\check{b}_{i'2}\}_{i'\in\mathcal{I}'}$ fulfil the analogous relations and generate an $\hat{H}$-module $(*\text{-})$subalgebra $\check{\mathcal{B}}_2 := D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(\mathcal{B}_2)$ isomorphic to $\mathcal{B}_\star$. Moreover, they fulfil the analogue of $(24)_2$.

Given an H-$(*\text{-})$module $\mathcal{M}$, $(\overline{\mathcal{F}}\triangleright^{\otimes 2}\mathcal{M}\otimes\mathcal{M}[[\lambda]])_\pm = (\mathcal{M}\otimes\mathcal{M})_\pm[[\lambda]]$ as vector spaces; both are also $\hat{H}$-$(*\text{-})$submodules of $\mathcal{M}\otimes\mathcal{M}[[\lambda]]$ (with involution (12)). Given an H-module $(*\text{-})$algebra $\mathcal{A}$, since $(\mathcal{A}\otimes\mathcal{A})_+$ is an H-module $(*\text{-})$subalgebra of $\mathcal{A}\otimes\mathcal{A}$, the *twisted completely symmetric tensor product algebra* $(\mathcal{A}\otimes\mathcal{A})_{+\star}$ is an $\hat{H}$-module $(*\text{-})$subalgebra of $(\mathcal{A}\otimes\mathcal{A})_\star$. It is easy to see that for any $b, c \in \mathcal{A}[[\lambda]]$

$$b_1\star c_2 + b_2\star c_1 = \sum_I \big[\overline{\mathcal{F}}_I^{(1)}\triangleright b\otimes\overline{\mathcal{F}}_I^{(2)}\triangleright c + \overline{\mathcal{F}}_I^{(2)}\triangleright c\otimes\overline{\mathcal{F}}_I^{(1)}\triangleright b\big] \in V[(\mathcal{A}\otimes\mathcal{A})_+][[\lambda]], \tag{38}$$

and actually the underlying vector space $V[(\mathcal{A}\otimes\mathcal{A})_{+\star}] = V[(\mathcal{A}\otimes\mathcal{A})_+][[\lambda]]$ is the linear span of elements of this form. If $\mathcal{A}$ can be defined in terms of generators a_i and polynomial relations, we obtain a basis of $V[(\mathcal{A}\otimes\mathcal{A})_{+\star}]$ letting b, c run over a basis $\{\hat{P}_J\}_{J\in\mathcal{J}}$ of $V(\mathcal{A}_\star) = V(\mathcal{A})[[\lambda]]$ consisting of $\star$-polynomials $\hat{P}_J(a_1\star, a_2\star, \ldots)$. If a $*$-algebra map $\sigma : H \mapsto \mathcal{A}$ of the type (26) exists, applying $D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{A}}}$ to (38) we find $\check{b}_1\check{c}_2 + \check{b}_2\check{c}_1$, which gives the realization of the elements of $(\mathcal{A}\otimes\mathcal{A})_{+\star}$ within $(\mathcal{A}\otimes\mathcal{A})[[\lambda]]$ in terms of polynomials in the $\check{a}_{i1}, \check{a}_{i2}$, since $\check{b}_1 = \hat{P}_J(\check{a}_{11}, \check{a}_{21}, \ldots)$, $\check{b}_2 = \hat{P}_J(\check{a}_{12}, \check{a}_{22}, \ldots)$, etc. The generalization to n-fold tensor products is straightforward.

We conclude with some link to the literature. As said, a first example of a deforming map is on $\mathcal{A} = H$ itself with $\sigma = \mathrm{id}$ (we shall abbreviate $D_{\mathcal{F}} \equiv D_{\mathcal{F}}^{\mathrm{id}}$); this was introduced in [39]. In [4] it has been applied to $H = U\Xi$, where Ξ stands for the Lie algebra of infinitesimal diffeomorphisms of a manifold X. By (4), (31), (32), $D_{\mathcal{F}}$ is a Hopf $*$-algebra isomorphism between $(\hat{H}, *, \hat{\Delta}, \epsilon, \hat{S}, \mathcal{R})$ and $(H_\star, *_\star, \Delta_\star, \epsilon, S_\star, \mathcal{R}_\star)$ if one defines, as in [4],

$$\Delta_\star := \left(D_{\mathcal{F}}^{-1} \otimes D_{\mathcal{F}}^{-1}\right) \circ \hat{\Delta} \circ D_{\mathcal{F}}, \qquad S_\star := D_{\mathcal{F}}^{-1} \circ \hat{S} \circ D_{\mathcal{F}}, \qquad \mathcal{R}_\star = (D_{\mathcal{F}}^{-1} \otimes D_{\mathcal{F}}^{-1})(\mathcal{R}).$$

So the two are essentially the same Hopf $*$-algebra; $D_{\mathcal{F}}^{-1}$ can be seen just as a change of generators. $*$-modules and module $*$-algebras of the former are also of the latter, if one defines $g \triangleright_\star a := D_{\mathcal{F}}(g) \triangleright a$ for any $g \in V(\hat{H}) = V(H_\star)$[11]. If both $g, h \in \mathbf{g}$, then also $g \triangleright_\star h \in \mathbf{g}$; this defines a '$\star$-Lie bracket' $[g, h]_\star$ [4] making $\mathbf{g}$ a *Lie $\star$-algebra* (in the parlance of [4]), a *generalized algebra* (in the parlance of [38, 43]), or a *quantum Lie algebra* (in the parlance of many authors, e.g. [10, 52], based on the results of [61]). We point out an important difference w.r.t. [4]. There the twist was assumed to fulfil $\mathcal{F}^{*\otimes *} = (S \otimes S)(\mathcal{F}_{21})$, rather than $\mathcal{F}^{*\otimes *} = \mathcal{F}^{-1}$; then the $*$-structures of the two isomorphic deformed Hopf algebras had to be taken respectively as $*_{\mathcal{F}}, *$, where $g^{*_{\mathcal{F}}} := \beta g^* \beta^{-1}$, rather than $*, *_\star$ (there the two Hopf $*$-algebras were denoted as $(U\mathbf{g}^{\mathcal{F}}, \cdot, *_{\mathcal{F}}, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathcal{R})$ and $(U\mathbf{g}_\star, \star, *, \Delta_\star, \epsilon, S_\star, \mathcal{R}_\star)$).

In [28] we introduced suitable σ and $D_{\mathcal{F}}^{\sigma}$ for general H-covariant Heisenberg or Clifford algebras, what we shall recall in subsection 3.4. In the next subsection we introduce σ and $D_{\mathcal{F}}^{\sigma}$ on the algebra of differential operators on $\mathbb{R}^m$.

3.3. Twisting functional, differential, integral calculi on $\mathbb{R}^m$

Here we apply the $\star$-deformation procedure to the algebras of functions and of differential operators on $\mathbb{R}^m$, as well as to the integration over $\mathbb{R}^m$.

$X = \mathbb{R}^m$ is invariant under the Lie group IGL(m) of real inhomogeneous general linear transformations. We call igl(m) the Lie algebra of IGL(m). Any set of cartesian coordinates $x^1, \ldots, x^m$ of $\mathbb{R}^m$ together with the unit $\mathbf{1}$ spans a Uigl(m)-$*$-module $\mathcal{M}$, with $g \triangleright \mathbf{1} = \varepsilon(g)\mathbf{1}$; the derivatives $\partial_1, \ldots, \partial_m$ span the contragredient one $\mathcal{M}'$. We shall denote as $\mathcal{D}_p$ the Heisenberg algebra on $\mathbb{R}^m$ (sometimes this is called the Weyl algebra), i.e. the $*$-algebra with generators, the unit $\mathbf{1}$ and $x^1, \ldots, x^m, \partial_1, \ldots, \partial_m$, relations given by trivial commutators except $[\partial_j, x^i] = \mathbf{1}\delta^i_j$, $*$-structure given by $x^{i*} = x^i$, $\partial_j{}^* = -\partial_j$. $\mathbf{1}, x^1, \ldots, x^m$ generate the Abelian subalgebra $\mathcal{X}_p$ of polynomials in x^i. $\mathcal{D}_p$ is a Uigl(m)-module $*$-algebra with a Uigl(m)-module $*$-subalgebra $\mathcal{X}_p$. Setting for brevity $x^{m+1} \equiv \mathbf{1}$, on the generators the action can be expressed in the form[12]

$$g \triangleright x^h = \tau^k_h(g) x^k, \qquad g \triangleright \partial_i = \tau^i_j[S(g)]\partial_j, \qquad \text{with} \quad \tau^k_{m+1}(g) = \varepsilon(g)\delta^k_{m+1}, \tag{39}$$

where repeated indices are summed over $1, 2, \ldots, m+1$.

For any Lie subalgebra $\mathbf{g} \subseteq$ igl(m) $\mathcal{D}_p$ is a $U\mathbf{g}$-module $*$-algebra with a $U\mathbf{g}$-module $*$-subalgebra $\mathcal{X}_p$. Taking $\mathcal{A} = \mathcal{D}_p$, $H = U\mathbf{g}$ and choosing a twist $\mathcal{F}$, we can define $\hat{H}$-module $*$-algebras $\mathcal{D}_{p\star}, \mathcal{X}_{p\star}, \ldots$ with $\mathcal{D}_{p\star} \supset \mathcal{X}_{p\star}$, as well as the map $\wedge : \mathcal{D}_p[[\lambda]] \mapsto \mathcal{D}_{p\star}$, through the $\star$-deformation procedure described in the previous subsections. In the appendix we show that $\widehat{\mathcal{D}_p} \sim \mathcal{D}_{p\star}$ amounts to the algebra generated by the unit $\mathbf{1} \equiv \hat{x}^{m+1}$ and $\hat{x}^1, \ldots, \hat{x}^m, \hat{\partial}'_1, \ldots, \hat{\partial}'_m$, fulfilling

$$\begin{aligned} &g \triangleright \hat{x}^h = \tau^k_h(g)\hat{x}^k, && g \triangleright \hat{\partial}'_a = \tau^a_b[\hat{S}(g)]\hat{\partial}'_b, && \\ &\hat{x}^{a\hat{*}} = \tau^k_a[S(\beta)]\hat{x}^k, && \hat{\partial}'^{\hat{*}}_a = -\tau^a_k(\beta^{-1})\hat{\partial}'_k, && \\ &\hat{x}^a\hat{x}^b = R^{kh}_{ab}\hat{x}^h\hat{x}^k, && \hat{\partial}'_a\hat{\partial}'_b = R^{ab}_{kh}\hat{\partial}'_h\hat{\partial}'_k, && \hat{\partial}'_a\hat{x}^b = \mathbf{1}\delta^b_a + R^{ha}_{bk}\hat{x}^h\hat{\partial}'_k, \end{aligned} \tag{40}$$

[11] In fact, $g \triangleright_\star (g' \triangleright_\star a) = D_{\mathcal{F}}(g) \triangleright [D_{\mathcal{F}}(g') \triangleright a] = D_{\mathcal{F}}(g) D_{\mathcal{F}}(g') \triangleright a = D_{\mathcal{F}}(g \star g') \triangleright a = (g \star g') \triangleright_\star a$, as required.

[12] Relation $(39)_1$ is the definition of the representation τ of Uigl(m) on $\mathcal{M}$; $(39)_3$ follows from $g \triangleright x^{m+1} = \tau^k_{m+1}(g)x^k = \varepsilon(g)x^{m+1}$. From $\varepsilon(g) = \sum_I S(g^I_{(1)})g^I_{(2)}$ it follows $\varepsilon(g)\delta^i_h = \sum_I \tau^i_k[S(g^I_{(1)})]\tau^k_h(g^I_{(2)})$, and by $(39)_3$ in the sum the term with $k = m+1$ vanishes. The latter relation implies that $(39)_2$ is the transformation law needed to preserve the relations $[\partial_j, x^h] = \mathbf{1}\delta^h_j$, besides $[\partial_j, \partial_k] = 0$ and $\partial_j{}^* = -\partial_j$.

where $R^{ab}_{hk} = \left(\tau^a_h \otimes \tau^b_k\right)(\mathcal{R})$ and $R^{ab}_{k(m+1)} = \delta^a_k \delta^b_{m+1} = R^{ba}_{(m+1)k}$. One can easily check that

$$\sigma(g) := (g \triangleright x^h)\partial_h = \tau^k_h(g) x^k \partial_h, \qquad g \in \mathbf{g}, \tag{41}$$

determines a map $\sigma : H \mapsto \mathcal{D}_p$ of the type described in subsection 3.2, so we can also define $D^\sigma_{\mathcal{F}}$ and therefore also $\check{x}^i, \check{\partial}_i$. For $f \in \mathcal{X}_p$ (37) becomes $D^\sigma_{\mathcal{F}}[f(x)] = \hat{f}(\check{x})$ where $\check{x}^h := D^\sigma_{\mathcal{F}}(x^h)$; applying the pseudodifferential operator $D^\sigma_{\mathcal{F}}(f)$ to the constant function $\mathbf{1}$ one finds (with the help of (30), $g \triangleright \mathbf{1} = \varepsilon(g)\mathbf{1}$ and $(4)_2$)

$$[\wedge^{-1}\hat{f}](x) \equiv f(x) = D^\sigma_{\mathcal{F}}(f) \triangleright \mathbf{1} = \hat{f}(\check{x}) \triangleright \mathbf{1}. \tag{42}$$

Thus, (42) gives an alternative way to compute the restriction $\wedge^{-1} : \widehat{\mathcal{X}_p} \mapsto \mathcal{X}_p[[\lambda]]$. Using the methods of the previous subsections one can similarly $\star$-deform $\mathcal{X}^{\otimes n}_p$, $\mathcal{D}^{\otimes n}_p$ and find the corresponding $\hat{H}$-module $*$-algebra isomorphism $\hat{D}^{\sigma_n}_{\mathcal{F}} : \widehat{\mathcal{D}^{\otimes n}_p} \mapsto \mathcal{D}^{\otimes n}_p[[\lambda]]$ (with $\sigma_n = \sigma^{\otimes n} \circ \Delta^{(n)}$). The existence of the latter is consistent with the well-known fact [23] that all deformations of the Heisenberg algebra are 'trivial', i.e. can be reabsorbed into (formal) changes of generators. Denoting by $x^h_1, x^h_2, \ldots$ the generators $x^h \otimes \mathbf{1} \otimes \cdots, \mathbf{1} \otimes x^h \otimes \cdots, \ldots$ of $\mathcal{X}^{\otimes n}_p$ and by $\partial_{x^a_1} = \partial/\partial x^a_1$, $\partial_{x^a_2} = \partial/\partial x^a_2, \ldots$ the remaining generators of $\mathcal{D}^{\otimes n}_p$, the relations (generalizing (40)) which characterize $\widehat{\mathcal{D}^{\otimes n}_p}$ read

$$\begin{aligned}
&\hat{x}^{a\hat{*}}_i = \tau^k_a[S(\beta)]\hat{x}^k_i, \qquad \hat{\partial}'^{\hat{*}}_{x^a_i} = -\tau^a_k(\beta^{-1})\hat{\partial}'_{x^k_i},\\
&\hat{x}^a_i\hat{x}^b_j = R^{kh}_{ab}\hat{x}^h_j\hat{x}^k_i, \qquad \hat{\partial}'_{x^a_i}\hat{\partial}'_{x^b_j} = R^{ab}_{kh}\hat{\partial}'_{x^h_j}\hat{\partial}'_{x^k_i}, \qquad \hat{\partial}'_{x^a_i}\hat{x}^b_j = \mathbf{1}\delta^b_a\delta^i_j + R^{ha}_{bk}\hat{x}^h_j\hat{\partial}'_{x^k_i}.
\end{aligned} \tag{43}$$

Since working with $\mathcal{X}_p$ is not sufficient for quantum mechanics (QM) and QFT purposes ($\mathcal{X}_p$ has e.g. no integrable functions), one has to extend the $\star$-deformation to larger function spaces $\mathcal{X}$. One could start with some subalgebra $\mathcal{X}$ of $C^\infty(\mathbb{R}^m)$ closed under the action of derivatives and with the algebra (and $\mathcal{X}$-bimodule) $\mathcal{D}$ of smooth differential operators consisting of polynomials in $\partial_1, \ldots, \partial_m$ with (left, say) coefficients in $\mathcal{X}$; its only nontrivial basic commutation relations are

$$[\partial_h, f] = \partial_h \triangleright f \qquad f \in \mathcal{X}.$$

$\mathcal{D}$ is an H-module $*$-algebra, which we can take as our $\mathcal{A}$. $\mathcal{X}$ is the H-module $*$-subalgebra consisting of differential operators of order zero. Clearly $\mathcal{D} \subset \mathcal{E} := \mathrm{End}(\mathcal{X})$: each $D \in \mathcal{D}$ defines an endomorphism $D : f \in \mathcal{X} \mapsto D \triangleright f \in \mathcal{X}$. On $\mathcal{X}$ the $\mathbf{g}$-action is given by $g \triangleright f = (g \triangleright x^h)(\partial_h \triangleright f)$. Applying the $\star$-deformation procedure to $\mathcal{A} = \mathcal{D}$ one obtains $\hat{H}$-module $*$-algebras $\mathcal{D}_\star$, $\mathcal{X}_\star$ with $\mathcal{D}_\star \supset \mathcal{X}_\star$; these are clearly extensions of $\mathcal{D}_{p\star}$, $\mathcal{X}_{p\star}$ if $\mathcal{X} \supset \mathcal{X}_p$. The same can be done with tensor powers $\mathcal{X}^{\otimes n}$, $\mathcal{D}^{\otimes n}$.

The Riemann integral is defined using the volume form $\mathrm{d}\nu = \mathrm{d}^m x$ associated with the Euclidean metric of $\mathbb{R}^m$; both $\mathrm{d}\nu$ and $\int_X \mathrm{d}\nu(\mathrm{x})$ are invariant under the isometry group $G = ISO(m) \subset IGL(m)$, or equivalently under $H = U\mathbf{g}$:

$$\int_X \mathrm{d}\nu (g \triangleright f) = \epsilon(g) \int_X \mathrm{d}\nu\, f \quad \Rightarrow \quad \int_X \mathrm{d}\nu\, f(g \triangleright h) = \int_X \mathrm{d}\nu [S(g) \triangleright f] h. \tag{44}$$

Fixing an $\mathcal{F} \in (H \otimes H)[[\lambda]]$, (44) implies the $\hat{H}$-invariance of $\int_X \mathrm{d}\nu$, as well as

$$\begin{aligned}
&\int_X \mathrm{d}\nu\, h \star h' = \int_X \mathrm{d}\nu\, h(\beta^{-1} \triangleright h') = \int_X \mathrm{d}\nu [S(\beta^{-1}) \triangleright h] h'\\
&\int_X \mathrm{d}\nu\, h^{*\star} \star h' = \int_X \mathrm{d}\nu\, h^* h'
\end{aligned} \tag{45}$$

for the corresponding $\star$-product. Hence,

$$\overline{\int_X d\nu\, f} = \int_X d\nu\, f^{*_\star}, \qquad \int_X d\nu\, h \star h' = \int_X d\nu (w \triangleright h') \star h,$$
$$\int_X d\nu\, h^{*_\star} \star h \geqslant 0, \qquad \text{and} = 0 \qquad \text{iff } h \equiv 0, \tag{46}$$

where $w := S(\beta)\beta^{-1}$: the Riemann integral fulfils in general the *modified trace property* $(46)_2$ w.r.t. such $\star$-products. So $w \neq \mathbf{1}$ is an obstruction for the star-product to be strongly closed in the sense of Connes–Flato–Sterheimer [19]. Of course, for the moment (44)–(46) make sense (as formal power series in λ) if the derivatives of f, h, h' of all orders are well defined and integrable, e.g. if $f, h, h' \in \mathcal{X} = \mathcal{S}(\mathbb{R}^m)$ (the Schwarz space). (But, again, $\mathcal{X} = \mathcal{S}(\mathbb{R}^m)$ is not large enough for QM and QFT purposes.) Analogous relations hold for integration over n independent x-variables. In $(\mathcal{X}^{\otimes n})_\star$ we define integration over the jth set of coordinates x_j $(j = 1, \ldots, n)$ of $\mathcal{X}^{\otimes n}$ in the natural way, i.e. as in the first equality of the formula

$$\int_X d\nu_j [f(\mathrm{x}_j) \star \omega] := \left[\int_X d\nu_j\, f(\mathrm{x}_j)\right] \star \omega = \omega \star \int_X d\nu_j\, f(\mathrm{x}_j), \tag{47}$$

for any ω depending only on the x_k $(k \neq j)$; here $d\nu_j r := r d^m x_j$. The second equality holds (with and without $\star$) trivially, because the integral belongs to $\mathbb{C}[[\lambda]]$. It follows that (see the appendix)

$$\int_X d\nu_j \omega \star f(\mathrm{x}_j) = \omega \star \int_X d\nu_j\, f(\mathrm{x}_j), \tag{48}$$

namely the integration $\int_X d\nu_j$ '$\star$-commutes' with (i.e. may be moved beyond) ω. More generally, (47) and (48) hold in any $\hat{H}$-module $*$-algebra $\Phi_\star \supseteq (\mathcal{X}^{\otimes n})_\star$ (e.g. the field $\star$-algebras of the next sections) if $\omega \in \Phi_\star$ does not depend on x_j, ∂_{x_j}.

If we can define, in terms of generators and relations only, $\widehat{\mathcal{X}}, \widehat{\mathcal{D}}$ isomorphic to $\mathcal{X}_\star, \mathcal{D}_\star$ for $\mathcal{X}, \mathcal{D}$ large enough, and extend $\wedge$ to such $\mathcal{X}, \mathcal{D}$[13], then we can define an $\hat{H}$-invariant 'integration over $\hat{X}$' $\int_{\hat{X}} d\hat{\nu}(\hat{\mathrm{x}})$ by the requirement

$$\int_{\hat{X}} d\hat{\nu}\, \hat{f}(\hat{\mathrm{x}}) = \int_X d\nu\, f(\mathrm{x}). \tag{49}$$

Then (44), (46), the definition of $\int_{\hat{X}} d\hat{\nu}_j$ and (48) would take the form

$$\int_{\hat{X}} d\hat{\nu}[g \triangleright \hat{f}(\hat{\mathrm{x}})] = \epsilon(g) \int_{\hat{X}} d\hat{\nu}\, \hat{f}(\hat{\mathrm{x}}), \qquad \overline{\int_{\hat{X}} d\hat{\nu}\, \hat{f}(\hat{\mathrm{x}})} = \int_{\hat{X}} d\hat{\nu} [\hat{f}(\hat{\mathrm{x}})]^{\hat{*}},$$
$$\int_{\hat{X}} d\hat{\nu} \hat{h}(\hat{\mathrm{x}}) \hat{h}'(\hat{\mathrm{x}}) = \int_{\hat{X}} d\hat{\nu} [w \triangleright \hat{h}'(\hat{\mathrm{x}})] \hat{h}(\hat{\mathrm{x}}), \qquad \int_{\hat{X}} d\hat{\nu} [\hat{h}(\hat{\mathrm{x}})]^{\hat{*}} \hat{h}(\hat{\mathrm{x}}) \geqslant 0, \quad = 0 \text{ iff } \hat{h} \equiv 0, \tag{50}$$
$$\int_{\hat{X}} d\hat{\nu}_j [\hat{f}(\hat{\mathrm{x}}_j) \hat{\omega}] := \left[\int_{\hat{X}} d\hat{\nu}_j\, \hat{f}(\hat{\mathrm{x}}_j)\right] \hat{\omega}, \qquad \int_{\hat{X}} d\hat{\nu}_j \hat{\omega} \hat{f}(\hat{\mathrm{x}}_j) = \hat{\omega} \int_{\hat{X}} d\hat{\nu}_j\, \hat{f}(\hat{\mathrm{x}}_j)$$

(with any $\hat{\omega}$ not depending on $\hat{\partial}_{\mathrm{x}_j}, \hat{\mathrm{x}}_j$). If we forget (49), requiring directly (50) and $\mathbb{C}[[\lambda]]$-linearity should be sufficient to determine up to a normalization constant the integration functional $\int_{\hat{X}} d\hat{\nu}(\hat{\mathrm{x}}) : \hat{f} \in \widehat{\mathcal{X}} \mapsto \mathbb{C}[[\lambda]]$. As a matter of fact, such an approach has been used quite successfully even on $\mathbb{R}^m_q$ [27, 56], the deformation [25] of $\mathbb{R}^m$ covariant under the quantum group (i.e. the proper *quasitriangular* Hopf algebra) $U_q so(m)$.

[13] It is not known how to do this for the general $\star$-deformation. For entire functions f one idea could be to express their power series expansion in $x^1, \ldots, x^m$ as formal power series $\hat{f}$ in $x^1\star, \ldots, x^m\star$, but even then this seems to be in general a non-computable operation; the coefficients of nth degree $\star$-monomials may receive contributions from the coefficients all monomials of degree $n' \geqslant n$.

The formal procedure just sketched can be extended to a class of $\mathbf{g}$-symmetric (possibly Riemannian) algebraic variety $X \subset \mathbb{R}^m$, where $\mathbf{g}$ is a Lie subalgebra of igl(m). This will be developed elsewhere.

The relations/definitions presented in this subsection are formal, both because of formal λ-power series and because of unspecified $\widehat{\mathcal{X}}$. In the next subsection we present the specific examples of $\star$-deformations induced by Moyal twists and study why they should allow us to go beyond the formal level (this will be studied in more detail elsewhere).

3.3.1. Application to Moyal deformations. In the last 15 years Moyal spaces[14] have been the subject of intensive investigations for their potential physical relevance. Because of the simplicity of their twist $\mathcal{F}$, they are also particularly pedagogical models; moreover, the associated $\star$-products admit (section 3.3.1) non-perturbative (in λ) definitions in terms of Fourier transforms. Here we give a detailed description of them.

One chooses a real Lie subalgebra $\mathbf{g}$ of igl(m) containing all the generators iP_h $(h = 1, \ldots, m)$ of translations, and as a twist

$$\mathcal{F} \equiv \sum_I \mathcal{F}_I^{(1)} \otimes \mathcal{F}_I^{(2)} := \exp\left(\frac{\mathrm{i}}{2}\lambda\vartheta^{hk} P_h \otimes P_k\right) = \exp\left(\frac{\mathrm{i}}{2}\theta^{hk} P_h \otimes P_k\right). \tag{51}$$

Here ϑ^{hk} is a fixed real antisymmetric matrix, $\sum_I$ includes the λ-power series arising from the expansion of the exponential and as conventional we absorb the deformation parameter λ in $\theta^{hk} := \lambda\vartheta^{hk}$. Straightforward computations give $\beta := \sum_I \mathcal{F}_I^{(1)} S(\mathcal{F}_I^{(2)}) = \mathbf{1}$, whence $*_\star = *$, and $\hat{\Delta}(P_h) = \Delta(P_h)$, so that the Hopf P-subalgebra remains undeformed and equivalent to the Abelian translation group $\mathbb{R}^m$.

Since iP_h acts on $\mathcal{X}_p$ as $\partial_h = \partial/\partial x^h$, it is i$P_h \triangleright x^k = \delta_h^k$ and $\tau_k^l(\mathrm{i}P_h) = \delta_h^k\delta_{m+1}^l$. The $\star$-product (13)+(51) on $\mathcal{X}_p^{\otimes n}$ becomes

$$a(x_1, \ldots, x_n) \star b(x_1, \ldots, x_n) = \exp\left[\frac{\mathrm{i}}{2}\left(\sum_i \partial_{x_i}\right)\theta\left(\sum_j \partial_{y_j}\right)\right] \times \triangleright\, a(x_1, \ldots, x_n) b(y_1, \ldots, y_n)\Big|_{y=x} \tag{52}$$

(we abbreviate $p\theta q := p_h\theta^{hk}q_k$ for any m-vectors p, q), in particular $x_i^h \star x_j^k = x_i^h x_j^k + \mathrm{i}\theta^{hk}/2$, whereas $a \star \partial_{x_j^h} = a\partial_{x_j^h}$, $\partial_{x_j^h} \star a = \partial_{x_j^h} a$ for any $a \in \mathcal{D}_p^{\otimes n}$ (the differential calculus is not deformed). Hence, relations (43) for $\widehat{\mathcal{D}_p^{\otimes n}} \sim \left(\mathcal{D}_p^{\otimes n}\right)_\star$ take the form

$$\begin{array}{lll} \hat{x}_i^{h\hat{*}} = \hat{x}_i^h, & \hat{\partial}_{x_i^h}{}^{\hat{*}} = -\hat{\partial}_{x_i^h}, & \\ \left[\hat{x}_i^h, \hat{x}_j^k\right] = \mathbf{1}\mathrm{i}\theta^{hk}, & \left[\hat{\partial}_{x_i^h}, \hat{x}_j^k\right] = \mathbf{1}\delta_h^k\delta_j^i, & \left[\hat{\partial}_{x_i^h}, \hat{\partial}_{x_j^k}\right] = 0. \end{array} \tag{53}$$

Using (41), (35) one constructs the isomorphism $\hat{D}_{\mathcal{F}}^{\sigma_n} : \widehat{\mathcal{D}_p^{\otimes n}} \mapsto \mathcal{D}_p^{\otimes n}[[\lambda]]$; one finds

$$\check{x}_i^h = \hat{D}_{\mathcal{F}}^{\sigma_n}\left(\hat{x}_i^h\right) = x_i^h + \mathrm{i}\theta^{hk}\left[\frac{1}{2}\partial_{x_i^k} + \sum_{j>i}^n \partial_{x_j^k}\right], \qquad \check{\partial}_{x_i^h} = \hat{D}_{\mathcal{F}}^{\sigma_n}\left(\hat{\partial}_{x_i^h}\right) = \partial_{x_i^h}, \tag{54}$$

[14] In the literature these are also called *canonical*, or more often denoted by some combinations of the names of Weyl, Wigner, Grönewold, Moyal. This is due to the relation between canonical commutation relations and the $\star$-product (or the twisted product) of Weyl and Von Neumann, which in turn was used by Wigner to introduce the Wigner transform; Wigner's work led Moyal to define the so-called Moyal bracket $[f \stackrel{\star}{,} g] = f \star g - g \star f$; the $\star$-product in the position space (in the form of the asymptotic expansion of (52) with $x_i = x_j \equiv x$) first appeared in a paper by Grönewold.

where $\sigma_n = \sigma^{\otimes n} \circ \Delta^{(n)}$, in particular $\check{x}^h = x^h + \mathrm{i}\theta^{hk}\partial_k/2$, $\check{\partial}_h = \partial_h$ for $n = 1$ (the latter is a result appeared several times in the literature). Formally, $(45)_1$ and $\beta = \mathbf{1}$ also imply

$$\int \mathrm{d}^m x a \star b = \int \mathrm{d}^m x ab = \int \mathrm{d}^m x b \star a, \tag{55}$$

and similarly for multiple integrations.

For any subspace $\mathcal{X} \subset C^\infty(\mathbb{R}^m)$, the λ-power series involved in (52) is termwise well defined if $a, b \in \mathcal{X}^{\otimes n}$ and reduces to a finite sum if at least one out of a, b is in $\mathcal{X}_p^{\otimes n}$ (a polynomial); the determination of the largest $\mathcal{X}$ such that it has a positive radius of convergence r is a delicate issue, about which little is known [24] (but again such a $\mathcal{X}$ would be too small for QM and QFT purposes). On the other hand, exponentials certainly belong to such $\mathcal{X}$, e.g. $\mathrm{e}^{\mathrm{i}h\cdot x_i} \star \mathrm{e}^{\mathrm{i}k\cdot x_j} = \mathrm{e}^{\mathrm{i}(h\cdot x_i + k\cdot x_j - \frac{h\theta k}{2})}$ for all matrices $\theta = \lambda\vartheta$ $(r = \infty)$, and their $\star$-product is associative as a consequence of the cocycle condition (5). Therefore, if $a, b \in \mathcal{X}$ admit Fourier transforms $\tilde{a}, \tilde{b}$, then[15]

$$a(x_i) \star b(x_j) = \int d^m h \int d^m k\, \mathrm{e}^{\mathrm{i}(h\cdot x_i + k\cdot x_j - \frac{h\theta k}{2})} \tilde{a}(h)\tilde{b}(k). \tag{56}$$

As a matter of fact, (56) and its generalization to a, b depending on all the x_i can be used as a *more general definition* of a (associative) $\star$-product. If $n = 1$ $(i = j = 1)$, it is a well-defined function for $a, b \in L^1(\mathbb{R}^m) \cap \widetilde{L^1(\mathbb{R}^m)}$; it makes sense also for $b \in \mathcal{X} = \mathcal{S}(\mathbb{R}^m)$ and $a \in \mathcal{X}'$ (the space of tempered distributions; $\tilde{a}$ is then the Fourier transform in the distribution sense) as the (symbolic) integrand in the definition of the distribution $a_\star : b \in \mathcal{X} \mapsto \int \mathrm{d}^m x\, a \star b \in \mathbb{C}$, as the latter integral is well defined (and actually equal to $\int \mathrm{d}^m x\, a\, b$), but if $a \in \mathcal{X}_p$ or $a \in \mathcal{X}$, this integrand $a \star b$ is also a true (actually $C^\infty(\mathbb{R}^m)$) function; and similarly with a, b interchanged; and so on. If $n = 2$ and $i = 1$, $j = 2$, it makes sense even for $a, b \in \mathcal{X}'$, as defining a new distribution $a \otimes_\star b$ on $c(x_1, x_2) \in \mathcal{X} \otimes \mathcal{X}$, showing $\mathcal{X}' \otimes_\star \mathcal{X}' = \mathcal{X}' \otimes \mathcal{X}'$[16]. In other words, using Fourier transforms the $\star$-products $a \star b$ become well defined in the same sense as the products ab; this also holds for a, b depending on all the x_i. So these $\star$-products are enough to replace all the products used in ordinary QFT, with results reducing to the commutative ones for $\theta^{hk} = 0$.

Morally, to extend $\widehat{\mathcal{X}_p^{\otimes n}}, \widehat{\mathcal{D}_p^{\otimes n}}$ we could consider the Weyl form of (53):

$$\hat{U}_i^p \hat{U}_j^q = \hat{U}_j^q \hat{U}_i^p\, \mathrm{e}^{-\mathrm{i}p\theta q}, \qquad \hat{V}_i^y \hat{U}_j^q = \hat{U}_j^q \hat{V}_i^y\, \mathrm{e}^{\mathrm{i}y\cdot q\delta_{ij}}, \qquad \hat{V}_i^y \hat{V}_j^z = \hat{V}_j^z \hat{V}_i^y \tag{57}$$

(with $p, q, y, z \in \mathbb{R}^m$) obtained by setting $\hat{U}_i^p := \mathrm{e}^{\mathrm{i}p\cdot\hat{x}_i}$, $\hat{V}_i^y := \mathrm{e}^{y\cdot\hat{\partial}_{x_i}}$. The $\hat{U}_i^p, \hat{V}_i^y$ generate a C^*-algebra isomorphic to the canonical one generated by $U_i^p := \mathrm{e}^{\mathrm{i}p\cdot x_i}$, $V_i^y := \mathrm{e}^{y\cdot\partial_{x_i}}$: the isomorphism is given by the extension of (54) to these exponentials:

$$\hat{D}_{\mathcal{F}}^{\sigma_n}\left(\hat{V}_i^y\right) = \mathrm{e}^{y\cdot\check{\partial}_{x_i}} = V_i^y, \qquad \hat{D}_{\mathcal{F}}^{\sigma_n}\left(\hat{U}_i^p\right) = \mathrm{e}^{\mathrm{i}p\cdot\check{x}_i} = U_i^p V_i^{\theta p/2} \prod_{j>i} V_j^{\theta p}. \tag{58}$$

The spaces of functions on $\mathbb{R}^m$ that one needs for QM and QFT (space of test functions like $\mathcal{X} = \mathcal{S}(\mathbb{R}^m)$, of square integrable functions $\mathcal{L}^2 \equiv \mathcal{L}^2(\mathbb{R}^m)$, space of distributions $\mathcal{X}'$, etc, and their tensor powers) all admit suitably generalized notions of the Fourier transformation

[15] (56) has the series (52) as a formal power expansion; see [24] for the conditions under which the latter is in fact an asymptotic expansion.

[16] Actually, for $i = j$ and some $a, b \in \mathcal{X}'$ it may even happen that (56) is ill-defined for $\theta^{\mu\nu} = 0$, but well defined [34] (and thus 'regularized') for $\theta^{hk} \neq 0$. For instance, for $a(x) = \delta^m(x) = b(x)$ and invertible θ one easily finds $a(x_i) \star b(x_j) = (\pi^m \det\theta)^{-1} \exp[2\mathrm{i}x_j\theta^{-1}x_i]$; in particular for $i = j$ the exponential becomes 1 by the antisymmetry of θ^{-1}, and one finds a diverging constant as $\det\theta \to 0$, cf [24, 34]. In [34] the largest algebra of distributions for which the $\star$-product is well defined and associative was determined.

(Fourier, Fourier–Plancherel, Fourier for distributions), and the generic element a of each of them can be expressed in terms of an anti-Fourier transform

$$a(x_1,\ldots,x_n)=\int \mathrm{d}^m q_1\cdots\int \mathrm{d}^m q_n\, \mathrm{e}^{\mathrm{i}q_1\cdot x_1}\cdots \mathrm{e}^{\mathrm{i}q_n\cdot x_n}\tilde{a}(q_1,\ldots,q_n), \tag{59}$$

where the symbols $\tilde{a}$ respectively belong to $\widetilde{\mathcal{X}^{\otimes n}}=\mathcal{X}^{\otimes n}$, $\widetilde{\mathcal{L}^{2\otimes n}}=\mathcal{L}^{2\otimes n}$, $\widetilde{\mathcal{X}'^{\otimes n}}=\mathcal{X}'^{\otimes n}$. We correspondingly define $\widehat{\mathcal{X}^{\otimes n}}$, $\widehat{\mathcal{L}^{2\otimes n}}$, $\widehat{\mathcal{X}'^{\otimes n}}$ as the spaces of objects of the form

$$\hat{a}(\hat{x}_1,\ldots,\hat{x}_n)=\int \mathrm{d}^m q_1\cdots\int d^m q_n\, \mathrm{e}^{\mathrm{i}q_1\cdot \hat{x}_1}\cdots \mathrm{e}^{\mathrm{i}q_n\cdot \hat{x}_n}\tilde{a}(q_1,\ldots,q_n). \tag{60}$$

By formally applying (42) to $\hat{U}^p(\hat{x})=\mathrm{e}^{\mathrm{i}p\cdot\hat{x}}$ one finds

$$[\wedge^{-1}\hat{U}^q](x)=\mathrm{e}^{\mathrm{i}q\cdot\check{x}}\triangleright\mathbf{1}=\mathrm{e}^{\mathrm{i}q\cdot(x+\frac{\mathrm{i}}{2}\theta\partial)}\triangleright\mathbf{1}=\mathrm{e}^{\mathrm{i}q\cdot x}=U^q(x)$$

(here $(\theta\partial)^h=\theta^{hk}\partial_k$), or equivalently $\wedge(U^q)=\hat{U}^q$, and, by iterated application of (21),

$$\wedge^n\left(U_1^{q_1}\ldots U_n^{q_n}\right)=\hat{U}_1^{q_1}\ldots\hat{U}_n^{q_n}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{i<j}q_i\theta q_j}. \tag{61}$$

We can *define* the extensions of $\wedge^n$ as maps from $\mathcal{X}^{\otimes n},\mathcal{L}^{2\otimes n},\mathcal{X}'^{\otimes n}$ respectively to $\widehat{\mathcal{X}^{\otimes n}}$, $\widehat{\mathcal{L}^{2\otimes n}}$, $\widehat{\mathcal{X}'^{\otimes n}}$ by linearity w.r.t. the 'generalized basis' $\{\hat{U}_1^{q_1}\ldots\hat{U}_n^{q_n}\}$:

$$[\wedge^n(a)](\hat{x}_1,\ldots,\hat{x}_n):=\int \mathrm{d}^m q_1\cdots\int \mathrm{d}^m q_n\, e^{iq_1\cdot\hat{x}_1}\cdots \mathrm{e}^{\mathrm{i}q_n\cdot\hat{x}_n}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{i<j}q_i\theta q_j}\tilde{a}(q_1,\ldots,q_n); \tag{62}$$

in the latter case this extends the linear map $\wedge^n:\mathcal{X}_p^{\otimes n}\mapsto\widehat{\mathcal{X}_p^{\otimes n}}$ defined in terms of the polynomial equation (20). For $n=1$ (62) is nothing but the well-known Weyl transformation. We can trivially extend $\wedge^n$ to the rest of $\mathcal{D}$ by setting $\wedge^n\left(V_i^p\right)=\hat{V}_i^p$, $\wedge^n\left(\partial_{x_i^h}\right)=\hat{\partial}_{x_i^h}$.

Another peculiar feature of Moyal spaces is that in $\widehat{\mathcal{X}^{\otimes n}}$ there is essentially *only one* set of noncommuting coordinates $(\hat{X}^h)$. Let $w_i\in\mathbb{R}$ with $\sum_{i=1}^n w_i=1$. An alternative set of real generators of both $\mathcal{X}^{\otimes n}$ and $(\mathcal{X}^{\otimes n})_\star$ is

$$\xi_i^h:=x_{i+1}^h-x_i^h,\quad i=1,\ldots,n-1,\qquad X^h:=\textstyle\sum_{i=1}^n w_i x_i^h \tag{63}$$

(one simplest choice is $X^h=x_1^h$). All ξ_i^h are translation invariant and therefore $\star$-commute with all the x_j^k, whereas X^h are not and fulfil $[X^h\stackrel{\star}{,}X^k]=\mathbf{1}\mathrm{i}\theta^{hk}$, whence

$$\left[\hat{\xi}_i^h,\hat{x}_j^k\right]=0,\qquad [\hat{X}^h,\hat{X}^k]=\mathbf{1}\mathrm{i}\theta^{hk}. \tag{64}$$

Therefore, $\hat{\xi}_i^h$ generate a polynomial algebra $\mathcal{X}_p^{n-1}$ isomorphic to $\mathcal{X}_p^{\otimes(n-1)}$, whereas $\hat{X}^h$ generate a polynomial algebra $\widehat{\mathcal{X}}_{pX}$ isomorphic to $\widehat{\mathcal{X}_p}$, and $\widehat{\mathcal{X}_p^{\otimes n}}\sim\mathcal{X}_p^{n-1}\otimes\widehat{\mathcal{X}}_{pX}$. Reasoning as above one can extend such a 'decoupling' to $\widehat{\mathcal{X}^{\otimes n}}$, $\widehat{(\mathcal{L}^2)^{\otimes n}}$, $\widehat{\mathcal{X}'^{\otimes n}}$. It is immediate to check that the map $\wedge^n:\mathcal{X}'^{\otimes n}\mapsto\widehat{\mathcal{X}'^{\otimes n}}$ is drastically simplified if we express the generic $f\in\mathcal{X}'^{\otimes n}$ as a function of X,ξ_i, $f(x_1,\ldots,x_n)=\sum_I f_X^I(X)f_\xi^I(\xi_1,\ldots,\xi_{n-1})$:

$$\widehat{f_X^I f_\xi^I}=\widehat{f_X^I}f_\xi^I. \tag{65}$$

3.4. Twisting Heisenberg/Clifford algebras $\mathcal{A}^\pm$

Here we $\star$-deform the Heisenberg algebra $\mathcal{A}^+$ (resp. Clifford algebra $\mathcal{A}^-$) associated with a species of bosons (resp. fermions) and its transformation properties.

We start fixing the notation while recalling the construction of $\mathcal{A}^\pm$ and how the transformation properties of $\mathcal{A}^\pm$ are induced from the one-particle ones. We describe the quantum system abstractly (i.e. in terms of bra, kets, abstract operators) and assume that a

Lie group G (the 'active transformations') is unitarily implemented on the Hilbert space of the system. The action of $U\mathbf{g}$ will be defined on a dense subspace; in particular, on the one-particle sector the action of $U\mathbf{g}$ will be defined on a pre-Hilbert space $\mathcal{H}$. Its closure $\overline{\mathcal{H}}$ is the one-particle Hilbert space. We call $\rho : H \hookrightarrow \mathcal{O} := \mathrm{End}(\mathcal{H})$ the $*$-algebra map such that $g \triangleright s = \rho(g)s$ for $s \in \mathcal{H}$. The compatibility condition $g \triangleright (Os) = \sum_I \big(g^I_{(1)} \triangleright O\big) g^I_{(2)} \triangleright s$ induces on $\mathcal{O}$ an H-module $*$-algebra structure (that of the adjoint action). The actions on $\mathcal{H}, \mathcal{O}$ are thus defined by

$$g \triangleright s = \rho(g)s, \qquad g \triangleright O = \sum_I \rho\big(g^I_{(1)}\big) O \rho\big[S\big(g^I_{(2)}\big)\big]. \tag{66}$$

The transformation of multi-particle systems, i.e. of $s \in \mathcal{H}^{\otimes n}$ and $O \in \mathcal{O}^{\otimes n}$, is obtained replacing ρ in (66) by $\rho^{(n)} := \rho^{\otimes n} \circ \Delta^{(n)}$. For all $s, v \in \mathcal{H}^{\otimes n}$ $\langle v, g \triangleright s\rangle = \langle g^* \triangleright v, s\rangle$. The pre-Hilbert space $\mathcal{H}^{\otimes n}_+$ of n-boson (resp. $\mathcal{H}^{\otimes n}_-$ of n-fermion) states is an H-$*$-submodule of $\mathcal{H}^{\otimes n}$, whereas $\mathcal{O}^{\otimes n}_+$ is an H-module $*$-subalgebra of $\mathcal{O}^{\otimes n}$; its elements map each of $\mathcal{H}^{\otimes n}_+, \mathcal{H}^{\otimes n}_-$ into itself. $\rho^{(n)}(H)$ is a module $*$-subalgebra of $\mathcal{O}^{\otimes n}_+$.

Let $\{e_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ be an orthonormal basis of $\overline{\mathcal{H}}$. For any $i_1, i_2, \ldots, i_n \in \mathbb{N}$ we denote

$$e^{\pm}_{i_1,i_2,\ldots,i_n} := N\mathcal{P}^{n\, j_1 j_2\cdots j_n}_{\pm i_1 i_2\cdots i_n}\big(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n}\big) \in \mathcal{H}^{\otimes n}_{\pm}, \tag{67}$$

where N is a normalization factor and $\mathcal{P}^n_{\pm}$ is the completely (anti)symmetric projector on $\mathcal{H}^{\otimes n}$. An orthonormal basis $\mathcal{B}^n_+$ (resp. $\mathcal{B}^n_-$) of $\overline{\mathcal{H}}^n_+$ (resp. $\overline{\mathcal{H}}^n_-$) is the set of the vectors (67) with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$ (resp. $i_1 < i_2 < \cdots < i_n$). As is known, each of the latter is more conveniently characterized by the sequence of occupation numbers $n_j \geqslant 0$; the integer n_j counts for how many h it occurs, $i_h = j$:

$$|n_1, n_2, \ldots\rangle := e^{\pm}_{i_1,i_2,\ldots,i_n}.$$

Clearly $\sum_{j=1}^{\infty} n_j = n$. Up to a phase $N = \sqrt{n!/\prod_{j=1}^{\infty} n_j!}$ (with $0! = 1$). For n identical fermions it can be only $n_j = 0, 1$, and $N = \sqrt{n!}$. Denoting as Ψ_0 the vacuum state, let

$$\mathcal{H}^{\infty}_{\pm} := \big\{\text{finite sequences } (s_0, s_1, s_2, \ldots) \in \mathbb{C}\Psi_0 \oplus \mathcal{H} \oplus \mathcal{H}^2_{\pm} \oplus \cdots\big\}$$

(finite sequence means that there exists an integer $l \geqslant 0$ such that $s_n = 0$ for all $n \geqslant l$). $\mathcal{H}^{\infty}_{\pm}$ is itself an H-$*$-module (we assume the vacuum to be H-invariant, $g \triangleright \Psi_0 = \varepsilon(g)\Psi_0$). The Fock space is defined as the closure $\overline{\mathcal{H}}^{\infty}_{\pm}$ of $\mathcal{H}^{\infty}_{\pm}$; it consists of sequences $(s_0, s_1, s_2, \ldots) \in \mathbb{C}\Psi_0 \oplus \overline{\mathcal{H}} \oplus \overline{\mathcal{H}}^2_{\pm} \oplus \cdots$ with finite norm. Let $\mathcal{B}^0_{\pm} := \{\Psi_0\}$. The set $\mathcal{B}^{\infty}_{\pm} := \cup_{n=0}^{\infty}\mathcal{B}^n_{\pm}$ is an orthonormal basis of the Fock space. It is the set of $|n_1, n_2, \ldots\rangle$ with $\sum_{j\in\mathbb{N}} n_j < \infty$. Creation and annihilation operators for bosons are defined by

$$a^+_i|\ldots, n_i, \ldots\rangle := \sqrt{n_i + 1}\,|\ldots, n_i + 1, \ldots\rangle, \qquad a^i|n_1, n_2, \ldots\rangle := \sqrt{n_i}\,|\ldots, n_i - 1, \ldots\rangle,$$

and for fermions by

$$a^+_i|\ldots, n_i, \ldots\rangle := (-1)^{s_i}(1 - n_i)\,|\ldots, n_i + 1, \ldots\rangle,$$
$$a^i|\ldots, n_i, \ldots\rangle := (-1)^{s_i} n_i\,|\ldots, n_i - 1, \ldots\rangle,$$

where $s_i := \sum_{j=1}^{i-1} n_j$. They fulfil the canonical (anti)commutation relations (CCR)

$$[a^i, a^j]_{\mp} = 0, \qquad \big[a^+_i, a^+_j\big]_{\mp} = 0, \qquad \big[a^i, a^+_j\big]_{\mp} = \delta^i_j\mathbf{1}. \tag{68}$$

The 'number-of-particles' $\mathbf{n} := a^+_i a^i$ (an infinite sum over i) is a densely defined operator on $\overline{\mathcal{H}}^{\infty}_{\pm}$ (a dense domain being $\mathcal{H}^{\infty}_{\pm}$) with a nonnegative discrete spectrum (actually $\{0, 1, 2, \ldots\}$, $\overline{H}^{\otimes n}_{\pm}$ being the eigenspace with eigenvalue n). As is known, this property, or alternatively the property $a^i\Psi_0 = 0$, uniquely characterizes (up to unitary equivalences) the Fock space representation among all irreducible representations of the Heisenberg (resp. Clifford) $*$-algebra $\mathcal{A}^{\pm}$ generated by a^i, a^+_i, $\mathbf{1}$ fulfilling (68). The H-$*$-module and the $\mathcal{A}$-module structures of $\mathcal{H}^{\infty}_{\pm}$ induce an H-module $*$-algebra structure on $\mathcal{A}^{\pm}$ (that of the adjoint action) through the compatibility requirement

$$g \triangleright (cs) = \sum_I \big(g^I_{(1)} \triangleright c\big) g^I_{(2)} \triangleright s, \qquad \forall\, g \in H, \quad c \in \mathcal{A}^{\pm}, \quad s \in \mathcal{H}^{\infty}_{\pm}. \tag{69}$$

Since $g \triangleright \Psi_0 = \varepsilon(g)\Psi_0$, a^+_i, a^i transform as $e_i = a^+_i \Psi_0$ and $\langle e_i, \cdot\rangle = \langle \Psi_0, a^i \cdot\rangle$ respectively:

$$g \triangleright e_i = \rho^j_i(g) e_j \quad \Rightarrow \quad g \triangleright a^+_i = \rho^j_i(g) a^+_j, \qquad g \triangleright a^i = \rho^{\vee}{}^j_i(g) a^j = \rho^i_j[S(g)] a^j \tag{70}$$

($\rho^{\vee} = \rho^T \circ S$ is the contragredient of ρ). So $\{a^+_i\}$ and $\{a^i\}$ respectively span carrier spaces of the representations $\rho, \rho^{\vee}$ of H. $\mathcal{A}^{\pm}$ is an H-module $*$-algebra because (68) generate an H-invariant $*$-ideal [$\triangleright$ is extended to products using (10)].

After this warm-up, we apply the $\star$-product deformation procedure and obtain a $*_\star$-algebra $\mathcal{A}^{\pm}_\star$ with generators a^+_i, a^i. It is convenient to replace the a^i by the $a'^i := \rho^i_j(\beta) a^j = a^{+*_\star}_i$, as the latter transform after the twisted contragredient representation, $g \triangleright a'^i = \hat{\rho}^{\vee}{}^j_i(g) a'^j = \rho^i_j[\hat{S}(g)] a'^j$, and, with the $\hat{a}^+_i$, fulfil the commutation relations on the lhs of

$$\begin{aligned} a'^i \star a'^j &= \pm R^{ij}_{vu} a'^u \star a'^v, & \hat{a}'^i \hat{a}'^j &= \pm R^{ij}_{vu} \hat{a}'^u \hat{a}'^v, \\ a^+_i \star a^+_j &= \pm R^{vu}_{ij} a^+_u \star a^+_v, \qquad \Leftrightarrow & \hat{a}^+_i \hat{a}^+_j &= \pm R^{vu}_{ij} \hat{a}^+_u \hat{a}^+_v, \\ a'^i \star a^+_j &= \delta^i_j \mathbf{1}_{\mathcal{A}} \pm R^{ui}_{jv} a^+_u \star a'^v, & \hat{a}'^i \hat{a}^+_j &= \delta^i_j \mathbf{1}_{\hat{\mathcal{A}}} \pm R^{ui}_{jv} \hat{a}^+_u \hat{a}'^v; \end{aligned} \tag{71}$$

here $R := (\rho \otimes \rho)(\mathcal{R})$. Omitting $\star$-product symbols and putting a $\hat{}$ over $*$ and all generators we obtain the isomorphic $\hat{H}$-module $*$-algebra $\widehat{\mathcal{A}^{\pm}}$ generated by $\hat{a}^+_i$, $\hat{a}'^i$ fulfilling the relations on rhs(71), the $\hat{*}$-conjugation relations $\hat{a}^{+\hat{*}}_i = \hat{a}'^i = \rho^i_j(\beta)\hat{a}^j$ and

$$g \hat{\triangleright} \hat{a}^+_i = \rho^j_i(g) \hat{a}^+_j, \qquad g \hat{\triangleright} \hat{a}'^i = \hat{\rho}^{\vee}{}^j_i(g) \hat{a}'^j = \rho^i_j(\hat{S}(g)) \hat{a}'^j. \tag{72}$$

Such a general class of equivariantly deformed Heisenberg/Clifford algebras was introduced in [28][17]. Up to normalization of R the relations on rhs(71) are actually identical to the ones defining the (older) q-deformed Heisenberg algebras of [48, 49, 60], based on a quasitriangular $\mathcal{R}$ in (only) the *fundamental* representation of $H = U_q su(N)$ (i.e. $i, j, u, v \in \{1, \ldots, N\}$).

On the other hand, following [28], it is immediate to check that setting

$$\sigma(g) := \big(g \triangleright a^+_j\big) a^j = \rho^i_j(g) a^+_i a^j, \qquad g \in \mathbf{g}, \tag{73}$$

defines a Lie $*$-algebra homomorphism $\sigma : \mathbf{g} \to \mathcal{A}^{\pm}$. This is extended as a $*$-algebra homomorphism $\sigma : \hat{H} = H[[\lambda]] \to \mathcal{A}^{\pm}[[\lambda]]$ over $\mathbb{C}[[\lambda]]$ by setting $\sigma(\mathbf{1}_H) := \mathbf{1}$ (for $\mathbf{g} = su(2)$ σ is the well-known Jordan–Schwinger realization of $Usu(2)$); moreover σ fulfils (26). It is also immediately checked that $\sigma(g)\Psi_0 = \varepsilon(g)\Psi_0$. It follows that

$$\sigma(g)s = g \triangleright s \tag{74}$$

for any $g \in \hat{H} = H[[\lambda]]$, $s \in \mathcal{H}^{\infty}_{\pm}[[\lambda]]$ (see the appendix). Setting

$$g \hat{\triangleright} s := g \triangleright s = \sigma(g)s \tag{75}$$

makes $\mathcal{H}^{\infty}_{\pm}[[\lambda]]$ an $\hat{H}$-$*$-module. $\mathcal{A}^{\pm}[[\lambda]]$ endowed with a $\hat{\triangleright}$ defined as in (27) is a compatible $\hat{H}$-module $*$-algebra, in the sense

$$g \hat{\triangleright} (cs) = \sum_I \big(g^I_{(\hat{1})} \hat{\triangleright} c\big) g^I_{(\hat{2})} \hat{\triangleright} s, \qquad \forall g \in \hat{H}, \quad c \in \mathcal{A}^{\pm}[[\lambda]] \tag{76}$$

[17] In [28] $\hat{a}^+_i, \hat{a}'^i, \beta, \hat{S}, \hat{\triangleright}, \hat{\rho}, \check{a}^+_i, \check{a}'^i$ were respectively denoted as $\tilde{A}^+_i, \tilde{A}^i, \gamma^{-1}, S_h, \triangleright_h, \tilde{\rho}, A^+_i, A^i$.

(see the appendix). Under $\hat{\triangleright}$ a_i^+, a^i *do not* transform as $\hat{a}_i^+, \hat{a}^i$ in (72), but the elements

$$\check{a}_i^+ = D_{\mathcal{F}}^{\sigma}\big(a_i^+\big) =: \widehat{D}_{\mathcal{F}}^{\sigma}\big(\hat{a}_i^+\big), \qquad \check{a}'^i = D_{\mathcal{F}}^{\sigma}(a'^i) = D_{\mathcal{F}}^{\sigma}\big[\rho_j^i(\beta)a^j\big] =: \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{a}'^i) \tag{77}$$

do [28]. Moreover, the latter fulfil rhs(71) and $\check{a}'^i = \check{a}_i^{+*}$. In other words, the 'dressed operators' $\check{a}_i^+, \check{a}^i$ provide a *realization of* $\hat{a}_i^+, \hat{a}^i$ *within* $\mathcal{A}^{\pm}[[\lambda]]$. Summing up, (77) define an $\hat{H}$-module $*$-algebra isomorphism (over $\mathbb{C}[[\lambda]]$) $\widehat{D}_{\mathcal{F}}^{\sigma} : \widehat{\mathcal{A}^{\pm}} \leftrightarrow \mathcal{A}^{\pm}[[\lambda]]$[18]:

$$\widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c}\hat{d}) = \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c})\widehat{D}_{\mathcal{F}}^{\sigma}(\hat{d}), \quad \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c}^{\hat{*}}) = \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c})^*, \quad g\hat{\triangleright}\widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c}) = \widehat{D}_{\mathcal{F}}^{\sigma}(g\hat{\triangleright}\hat{c}). \tag{78}$$

Then the $\mathcal{A}^{\pm}[[\lambda]]$-$*$-module $\mathcal{H}_{\pm}^{\infty}[[\lambda]]$ becomes also an $\widehat{\mathcal{A}^{\pm}}$-$*$-module if one sets

$$\hat{c}s := \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c})s \tag{79}$$

for any $\hat{c} \in \widehat{\mathcal{A}^{\pm}}$, $s \in \mathcal{H}_{\pm}^{\infty}[[\lambda]]$. As $\hat{a}'^i\Psi_0 = \check{a}'^i\Psi_0 = 0$, one finds that this is a Fock-space-type representation of $\widehat{\mathcal{A}^{\pm}}$ on $\mathcal{H}_{\pm}^{\infty}[[\lambda]]$. $\hat{*}$ plays also the role of Hermitian conjugation w.r.t. the scalar product of $\mathcal{H}_{\pm}^{\infty}$, since

$$\langle \hat{c}s_1, s_2\rangle \overset{(79)}{=} \big\langle \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c})s_1, s_2\big\rangle = \big\langle s_1, \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c})^*s_2\big\rangle \overset{(78)}{=} \big\langle s_1, \widehat{D}_{\mathcal{F}}^{\sigma}(\hat{c}^{\hat{*}})s_2\big\rangle \overset{(79)}{=} \langle s_1, \hat{c}^{\hat{*}}s_2\rangle \tag{80}$$

for any $\hat{c} \in \widehat{\mathcal{A}^{\pm}}$, $s_1, s_2 \in \mathcal{H}_{\pm}^{\infty}[[\lambda]]$. Also note that $\hat{a}_i^+\Psi_0 = \check{a}_i^+\Psi_0 = a_i^+\Psi_0 = e_i$. Finally, the analogue of the compatibility (76) (with $\hat{c}$ replacing c) holds.

In terms of $\star$-products the previous property becomes the first equality in

$$\langle cs_1, s_2\rangle = \langle s_1, c^{*_\star} \star s_2\rangle = \langle s_1, c^* s_2\rangle, \tag{81}$$

valid for any $c \in \mathcal{A}^{\pm}[[\lambda]]$: the Hermitian conjugation is $*$ if we multiply operators by the original product and $*_\star$ if we multiply them by the $\star$-product. An independent proof of this (see the appendix) relies on the relation

$$\langle \Psi_0, r^*t\Psi_0\rangle = \langle \Psi_0, r^{*_\star} \star t\Psi_0\rangle, \qquad \forall\, r, t \in \mathcal{A}^{\pm}[[\lambda]] \tag{82}$$

(this is the analogue of $(45)_2$) , which in turn is proved using the H-invariance of Ψ_0.

Choosing $\hat{c} = \hat{a}_{i_1}^+ \dots \hat{a}_{i_n}^+$ in definition (79) one obtains the first equality in the formula

$$\hat{a}_{i_1}^+ \cdots \hat{a}_{i_n}^+ \Psi_0 = \check{a}_{i_1}^+ \cdots \check{a}_{i_n}^+ \Psi_0 = \overline{F}^n{}_{i_1,\dots,i_n}^{j_1,\dots,j_n}\, a_{j_1}^+ \cdots a_{j_n}^+ \Psi_0 = a_{i_1}^+ \star \cdots \star a_{i_n}^+ \Psi_0 \in \mathcal{H}_{\pm}^n[[\lambda]]. \tag{83}$$

The second is easily proved using $(10)_3$ and the inverse of definition $(9)_1$. The third follows applying the vacuum to both sides of the identity $a_{i_1}^+ \star \dots \star a_{i_n}^+ = \overline{F}^n{}_{i_1,\dots,i_n}^{j_1,\dots,j_n}\, a_{j_1}^+ \dots a_{j_n}^+$, where $\overline{F}^n := \rho^{\otimes n}[(\mathcal{F}^n)^{-1}]$ (a unitary transformation of $\mathcal{H}^{\otimes n}$); this is an identity in $V(\mathcal{A}^{\pm})[[\lambda]]$ following from (13). A straightforward computation gives

$$\begin{aligned} \sigma(g)a_{i_1}^+ \cdots a_{i_n}^+ \Psi_0 &= [\rho^{(n)}(g)]_{i_1\dots i_n}^{j_1\dots j_n} a_{j_1}^+ \cdots a_{j_n}^+ \Psi_0, \\ \sigma(g)a_{i_1}^+ \star \cdots \star a_{i_n}^+ \Psi_0 &= [\hat{\rho}^{(n)}(g)]_{i_1\dots i_n}^{j_1\dots j_n} a_{j_1}^+ \star \dots \star a_{j_n}^+ \Psi_0, \end{aligned} \tag{84}$$

where $\hat{\rho}^{(n)}(g) = \rho^{\otimes n} \circ \hat{\Delta}^{(n)}(g) = F^n \rho^{(n)}(g)\overline{F}^n$. Also note that $a_i^+ \star a'^i = \check{a}_i^+\check{a}'^i = a_i^+a^i = \mathbf{n}$. Let $e'^{\pm}_{i_1,\dots,i_n}$ be the vectors (83) multiplied by $\sqrt{n!}/N$; a (non-orthonormal) basis $\mathcal{B}'^n_+$ of $\mathcal{H}_+^{\otimes n}[[\lambda]]$ (resp. $\mathcal{B}'^n_-$ of $\mathcal{H}_-^n[[\lambda]]$) over $\mathbb{C}[[\lambda]]$ consists of those with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$ (resp. $i_1 < i_2 < \cdots < i_n$).

The vectors (83) are up to now formal λ-power series (arising from the λ-power series expansions of $\overline{F}^n{}_{i_1,\dots,i_n}^{j_1,\dots,j_n}$) with coefficients in $\mathcal{H}_{\pm}^{\otimes n}$. If one can really define a 1-parameter continuous family of unitary operators $\lambda \in \mathbb{R} \mapsto \overline{F}^n(\lambda)$ (at least in a neighbourhood of

[18] The 'triviality' of any deformation $\widehat{\mathcal{A}^{\pm}}$ of any Heisenberg algebra $\mathcal{A}^{\pm}$, i.e. the existence of a isomorphism $\widehat{D}_{\mathcal{F}}^{\sigma} : \widehat{\mathcal{A}^{\pm}} \leftrightarrow \mathcal{A}^{\pm}[[\lambda]]$ of $*$-algebras in the form of a formal power series in λ which is the identity at order λ^0, is also a consequence [23] of the vanishing of its second cohomological group; this however does not give any such isomorphism explicitly.

$\lambda = 0$) such that the mentioned λ-power series are the asymptotic expansions of $\overline{F}^{n}{}^{j_1,\ldots,j_n}_{i_1,\ldots,i_n}$, then (83) are true vectors of $\mathcal{H}_{\pm}^{\otimes n}$ allowing a *representation of* $\widehat{\mathcal{A}^{\pm}}$ *on the ordinary (Bose–Fermi) Fock space* $\overline{\mathcal{H}}_{\pm}^{\infty}$; as a consequence, *no 'change of statistics' is needed to represent the deformed Heisenberg (resp. Clifford) algebra* $\widehat{\mathcal{A}^{\pm}}$. This occurs e.g. in all deformations based on 'Reshetikhin twists' [50], i.e. twists of the form $\mathcal{F} = \mathrm{e}^{\mathrm{i}\lambda f}$, where $f \in \mathbf{h} \otimes \mathbf{h}$ and $\mathbf{h}$ is a real Cartan subalgebra of $\mathbf{g}$; then choosing a basis e_i of $\mathcal{H}$ consisting of eigenvectors of $\mathbf{h}$ the unitary matrix $\overline{F}^{n}(\lambda)$ becomes diagonal with elements of modulus 1 (phases). This applies in particular to the Moyal twists (see sections 5, 6.1), where one uses a generalized basis of $\mathcal{H}$ consisting of eigenvectors of the translation Lie subalgebra $\mathbf{h}$. More generally, this will occur whenever $\hat{H}$ admits a non-perturbative (in λ) representation on $\mathcal{H}$ in one-to-one correspondence with that of H; then the matrix elements of $\overline{F}^{n}(\lambda)$ will be given by contractions of deformed and undeformed Clebsch–Gordan coefficients [11]. The latter fact is true even for the matrix elements of the generalized twist of some *quasitriangular* (but not triangular) Hopf algebras like the semi-simple Drinfel'd–Jimbo quantum groups (in a *rational* form), where the dependence on λ of the deformed Clebsch–Gordan coefficients can be concentrated in a *rational* dependence on the new parameter $q = \mathrm{e}^{\lambda} \in \mathbb{C}$, for generic q.

Does $\widehat{\mathcal{A}^{\pm}}$ admit other $*$-representations of Fock type which are not unitarily equivalent to $*$-representations of $\mathcal{A}^{\pm}$? Using standard arguments it is easy to see that the answer is negative in either characterization of the Fock representation. In fact, from the first characterization (the existence of a unique vacuum $\hat{\Psi}_0$: $\hat{a}'^i\hat{\Psi}_0 = 0$ for all i) and the commutation relations (71) it follows that the map

$$P(\hat{a}', \hat{a}^+)\hat{\Psi}_0 \mapsto P(\check{a}', \check{a}^+)\Psi_0$$

for all polynomial $P(\hat{a}', \hat{a}^+)$ preserves the scalar products, and therefore is unitary. As for the second, (71) imply that the nonnegative-definite, infinite sum over i $\hat{\mathbf{n}} := \hat{a}^+_i\hat{a}'^i = (\hat{a}'^i)^{\hat{*}}\hat{a}'^i$ fulfils $[\hat{\mathbf{n}}, \hat{a}'^i] = -\hat{a}'^i$, $\left[\hat{\mathbf{n}}, \hat{a}^+_i\right] = \hat{a}^+_i$; requiring that $\hat{\mathbf{n}}$ is densely defined and has a nonnegative discrete spectrum, in particular an eigenvector ν with eigenvalue $\nu \geqslant 0$, implies that a vacuum $\hat{\Psi}_0 \neq 0$ arises applying to ν a suitable monomial in the $\hat{a}'^i$.

4. Non-relativistic second quantization

4.1. Twisting quantum mechanics in configuration space

Dealing with the wave-mechanical description of a system of quantum particles in space X means that the state vectors s's are described by wavefunctions ψ's on X and the abstract operators by differential or more generally integral operators on the ψ's. For simplicity we stick to[19] $X = \mathbb{R}^3$, consider spinless particles and derive consequences from the covariance of the description first under the Euclidean group G (thought as a group of *active* space-symmetry transformations) and then under the whole Galilei group G'. We shall call x^a a set of Cartesian coordinates of $\mathbb{R}^3$. Going to the infinitesimal form, all elements $H = U\mathbf{g}$ will be well-defined differential operators e.g. on the pre-Hilbert space $\mathcal{S}(\mathbb{R}^3)$, so we can choose $\mathcal{X}$ as a dense subspace $\mathcal{X} \subseteq \mathcal{S}(\mathbb{R}^3)$ (to be specified later) and tailor the one-particle pre-Hilbert space $\mathcal{H}$ and the algebra of endomorphisms $\mathcal{O} := \mathrm{End}(\mathcal{H})$ as respectively isomorphic to $\mathcal{X}$ and $\mathcal{E} := \mathrm{End}(\mathcal{X})$, by definition; we shall call the isomorphisms $\kappa, \tilde{\kappa}$. (One reason why we do not identify $\mathcal{H}$ with $\mathcal{X}$ is that we wish to introduce a *realization* (i.e. representation) of the *same* state $s \in \mathcal{H}$ of the quantum system also by a noncommutative wavefunction.)

[19] Morally, one could apply such an approach to any Riemannian manifold X (playing the role of 'space') for which the first quantization is well defined and has a covariance Lie group G' containing as a subgroup the isometries of the spacetime $X' = \mathbb{R} \times X$.

Summarizing, there exists a (reference frame-dependent) $H = U\mathbf{g}$-*equivariant configuration space realization of* $\{\mathcal{H}, \mathcal{O}\}$ *on* $\{\mathcal{X}, \mathcal{E}\}$, i.e.

(1) there exists an H-equivariant, unitary transformation $\kappa : s \in \mathcal{H} \leftrightarrow \psi_s \in \mathcal{X}$, i.e.

$$g \triangleright \psi_s = \psi_{g\triangleright s}, \qquad \langle s|v\rangle = \int_X d\nu[\psi_s(\mathrm{x})]^*\psi_v(\mathrm{x}). \tag{85}$$

(2) $\kappa(Os) = \tilde{\kappa}(O)\kappa(s)$ for any $s \in \mathcal{H}$ defines an H-equivariant map $\tilde{\kappa} : \mathcal{O} \leftrightarrow \mathcal{E}$ (equivariance means $g \triangleright \tilde{\kappa}(O) = \tilde{\kappa}(g \triangleright O)$), and $\mathcal{D} \subset \mathcal{E}$ ($\mathcal{D}$ was defined in section 3.3). In particular $\tilde{\kappa}(q^a) = \mathrm{x}^a\cdot$, $\tilde{\kappa}(p^a) = -\mathrm{i}\hbar\frac{\partial}{\partial \mathrm{x}^a}$ on the canonical variables $\{q^a, p^a\}$.

This implies for a system of n distinct particles (resp. n bosons/fermions) on X:

(1) $\kappa^{\otimes n} : \mathcal{H}^{\otimes n} \leftrightarrow \mathcal{X}^{\otimes n}$ (resp. their restrictions $\kappa^n_\pm : \mathcal{H}^{\otimes n}_\pm \leftrightarrow \mathcal{X}^{\otimes n}_\pm$) are H-equivariant unitary transformations.
(2) $\tilde{\kappa}^{\otimes n} : \mathcal{O}^{\otimes n} \leftrightarrow \mathcal{E}^{\otimes n}$ (resp. its restriction $\tilde{\kappa}^n_+ : \mathcal{O}^{\otimes n}_+ \leftrightarrow \mathcal{E}^{\otimes n}_+$) are H-equivariant maps.

The maps $\kappa^n_\pm, \tilde{\kappa}^n_+$ define a (frame-dependent) $U\mathbf{g}$-*equivariant, commutative configuration space realization* of $\{\mathcal{H}^{\otimes n}_\pm, \mathcal{O}^{\otimes n}_+\}$ (the Hilbert space and algebra of observables of a system of n bosons/fermions) on $\{\mathcal{X}^{\otimes n}_\pm, \mathcal{E}^{\otimes n}_+\}$.

For any twist $\mathcal{F} \in (H \otimes H)[[\lambda]]$ and the associated $\star$-deformation one finds

$$\begin{aligned}\langle s, v\rangle &= \textstyle\int_X d\nu(\mathrm{x}_1)\cdots\int_X d\nu(\mathrm{x}_n)[\psi_s(\mathrm{x}_1,\ldots,\mathrm{x}_n)]^*\psi_v(\mathrm{x}_1,\ldots,\mathrm{x}_n)\\ &\overset{(45)}{=} \textstyle\int_X d\nu(\mathrm{x}_1)\cdots\int_X d\nu(\mathrm{x}_n)[\psi_s(\mathrm{x}_1,\ldots,\mathrm{x}_n)]^{*_\star}\star\psi_v(\mathrm{x}_1,\ldots,\mathrm{x}_n).\end{aligned} \tag{86}$$

If in addition $\mathcal{F}$ is such that one can define $\widehat{\mathcal{X}}, \widehat{\mathcal{D}}$ and the map $\wedge$ (see section 3.3), we introduce noncommutative wavefunctions $\hat{\psi} = \wedge^n(\psi)$. Then the previous equation becomes

$$\langle s, v\rangle = \int_{\hat{X}} d\hat{\nu}(\hat{\mathrm{x}}_1)\ldots\int_{\hat{X}} d\hat{\nu}(\hat{\mathrm{x}}_n)[\hat{\psi}_s(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n)]^{\hat{*}}\hat{\psi}_v(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n). \tag{87}$$

The map $\wedge^n : \psi_s \in \mathcal{X}^{\otimes n} \mapsto \hat{\psi}_s \in \widehat{\mathcal{X}^{\otimes n}}$ is thus unitary and $\hat{H}$-equivariant, $\widehat{g\triangleright\psi_s} = g \triangleright \hat{\psi}_s$; using (21) one finds $\wedge^n = \wedge^{\otimes n} \circ (\mathcal{F}^n\triangleright^{\otimes n})$.

The action of the symmetric group S_n on $\widehat{\mathcal{X}^{\otimes n}}, \widehat{\mathcal{E}^{\otimes n}}$ is obtained by 'pull-back' from that on $\mathcal{X}^{\otimes n}, \mathcal{E}^{\otimes n}$. A permutation $\tau \in S_n$ is represented on $\mathcal{X}^{\otimes n}, \widehat{\mathcal{X}^{\otimes n}}$ respectively by the permutation operator $\mathcal{P}_\tau$ and the 'twisted permutation operator' $\mathcal{P}^F_\tau = \wedge^n\mathcal{P}_\tau[\wedge^n]^{-1}$ (cf [31]). The completely (anti)symmetric projector $\mathcal{P}^n_\pm = (1/n!)\sum_{\tau\in S_n}\eta_\tau P_\tau$ ($\eta_\tau = -1$ if τ is an odd permutation and we consider $\mathcal{P}^n_-$, $\eta_\tau = 1$ otherwise) and its twisted counterpart $\mathcal{P}^{nF}_\pm := \wedge^n\mathcal{P}^n_\pm[\wedge^n]^{-1}$ respectively project $\mathcal{X}^{\otimes n}$ and $\widehat{\mathcal{X}^{\otimes n}}$ onto their subspaces $\mathcal{X}^{\otimes n}_\pm$ and $\wedge^n(\mathcal{X}^{\otimes n}_\pm)$, which are eigenspaces of $\mathcal{P}_\tau$ and $\mathcal{P}^F_\tau$ with the eigenvalue η_τ[20]. Thus, $\widehat{\mathcal{X}^{\otimes n}}, \widehat{\mathcal{E}^{\otimes n}}$ are (anti)symmetric up to the similarity transformation $\wedge^n$ (cf [31]). As an example, in section 5 we exhibit $\mathcal{P}^{2,F}_\pm$ on the Moyal space.

Let $\hat{\kappa}^n := \wedge^n\kappa^{\otimes n}$, $\widehat{\tilde{\kappa}}^n(\cdot) := \wedge^n[\tilde{\kappa}^{\otimes n}(\cdot)][\wedge^n]^{-1}$. The restrictions $\hat{\kappa}^n_\pm := \hat{\kappa}^n \restriction_{\mathcal{H}^{\otimes n}_\pm}$, $\widehat{\tilde{\kappa}}^n_+ := \widehat{\tilde{\kappa}}^n \restriction_{\mathcal{O}^{\otimes n}_+}$ define a (frame-dependent) $\hat{H}$-*equivariant, noncommutative configuration space realization* of $\{\mathcal{H}^{\otimes n}_\pm, \mathcal{O}^{\otimes n}_+\}$ on $\{\widehat{\mathcal{X}^{\otimes n}_\pm}, \widehat{\mathcal{E}^{\otimes n}_+}\}$.

[20] One can easily show that $\mathcal{P}^F_\tau, \mathcal{P}^{nF}_\pm$ depend on $\mathcal{F}$ through the $\mathcal{R}$-matrix only. If $\mathcal{P}_\tau = \prod_{\{(hk)\}}\mathcal{P}_{hk}$ is a decomposition of $\mathcal{P}_\tau$ as a product of transpositions ($\mathcal{P}_{hk}$ stands for the transposition of the hth and kth tensor factors), $\mathcal{P}^F_\tau$ can be decomposed as $\mathcal{P}^F_\tau = \prod_{\{(hk)\}}\mathcal{P}_{hk}\mathcal{R}_{hk}\triangleright^{\otimes n}$, where $\mathcal{R}_{hk} = \sum_I \mathbf{1}^{\otimes(h-1)}\otimes\mathcal{R}^{(1)}_I\otimes\mathbf{1}^{\otimes(k-h-1)}\otimes\mathcal{R}^{(2)}_I\otimes\mathbf{1}^{\otimes(n-k)}$. If e.g. $n = 2$ and $\tau = (21)$, we find $\mathcal{P}_\tau[\psi_s(\mathrm{x}_1)\psi_v(\mathrm{x}_2)] = \psi_v(\mathrm{x}_1)\psi_s(\mathrm{x}_2)$, whereas

$$\mathcal{P}^F_\tau[\hat{\psi}_s(\hat{\mathrm{x}}_1)\hat{\psi}_v(\hat{\mathrm{x}}_2)] = \sum_{I,I'}[\mathcal{F}^{(1)}_I\overline{\mathcal{F}}^{(2')}_{I'}\triangleright\hat{\psi}_v(\hat{\mathrm{x}}_1)][\mathcal{F}^{(2)}_I\overline{\mathcal{F}}^{(1')}_{I'}\triangleright\hat{\psi}_s(\hat{\mathrm{x}}_2)] = \sum_I[\mathcal{R}^{(1)}_I\triangleright\hat{\psi}_v(\hat{\mathrm{x}}_1)][\mathcal{R}^{(2)}_I\triangleright\hat{\psi}_s(\hat{\mathrm{x}}_2)]. \tag{88}$$

4.2. Twisting quantum fields in the Schrödinger picture

Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{X}$, $\{e_i\}_{i\in\mathbb{N}}$ the corresponding basis of $\mathcal{H}$ and φ_i $(=\kappa(e_i))$, a_i^+, a^i the associated wavefunction, creation and annihilation operators, respectively. We adopt the dual $\mathcal{X}'$ of $\mathcal{X}$ as space of distributions. The non-relativistic field operator φ in the *Schrödinger picture* and its Hermitian conjugate φ^* defined in (1) are operator-valued distributions fulfilling the canonical (anti)commutation relations

$$[\varphi(\mathrm{x}),\varphi(\mathrm{y})]_\mp=\text{h.c.}=0,\qquad [\varphi(\mathrm{x}),\varphi^*(\mathrm{y})]_\mp=\varphi_i(\mathrm{x})\varphi_i^*(\mathrm{y})=\delta(\mathrm{x}-\mathrm{y}) \tag{89}$$

($\mp$ for bosons/fermions; infinite sum over i; $\varphi_i^*\equiv\overline{\varphi_i}$). One needs to work with polynomials of arbitrarily high degree n in φ,φ^* evaluated at independent points $\mathrm{x}_1,\ldots,\mathrm{x}_n$; the generic normal ordered monomial of degree n will read

$$\varphi^*(\mathrm{x}_1)\cdots.\varphi^*(\mathrm{x}_m)\varphi(\mathrm{x}_{m+1})\cdots\varphi(\mathrm{x}_n) \tag{90}$$

with $0\leqslant m\leqslant n$. We shall call the linear span of all such monomials (for all m,n) the *field $*$-algebra* Φ. This is a subalgebra of the tensor product algebra $\Phi^e=\mathcal{A}^\pm\otimes\left(\bigotimes_{i=1}^\infty\mathcal{X}'\right)$, where the first, second,... tensor factor $\mathcal{X}'$ refers to the dependence of the distribution on $\mathrm{x}_1,\mathrm{x}_2,\ldots$ respectively (by definition, the dependence of the monomial (90) on x_h is trivial for $h>n$). The CCR (68) of $\mathcal{A}^\pm$ are the only nontrivial commutation relations in Φ^e. Relations (89) hold with any $\mathrm{x},\mathrm{y}\in\{\mathrm{x}_1,\mathrm{x}_2,\ldots\}$.

The key property is that φ,φ^* are basis independent, i.e. *invariant under the group $U(\infty)$ of unitary transformations of $\mathcal{H}$ applied to $\{e_i\}_{i\in\mathbb{N}}$*, in particular under the subgroup G of active space-symmetry transformations (transformations of the states e_i obtained by translations or rotations of the one-particle system), or (in infinitesimal form) *under $U\mathbf{g}$*:

$$g\triangleright\varphi(\mathrm{x})=\epsilon(g)\varphi(\mathrm{x}),\qquad g\triangleright\varphi^*(\mathrm{x})=\epsilon(g)\varphi^*(\mathrm{x}). \tag{91}$$

Instead, φ_i, a_i^+, e_i transform after the same nontrivial representation ρ of $U(\infty)$, $\varphi_i^*, a^i, \langle e_i,\cdot\rangle$ after the contragredient $\rho^\vee$. Altogether, Φ^e is a huge $U\mathbf{g}$-module (and also $Uu(\infty)$-module) $*$-algebra.

As a consequence, if we extend the $\star$-deformation of the previous sections to $Uu(\infty)$ and Φ^e, we obtain the Hopf algebra $\widehat{Uu(\infty)}$ and the $\widehat{U\mathbf{g}}$-module (and also $\widehat{Uu(\infty)}$-module) $*$-algebra $\Phi^e_\star$. We find $\varphi^{*_\star}(\mathrm{x})=\varphi^*(\mathrm{x})$ by (12), (91) and $\epsilon(\beta)=1$. Moreover, by (15) the $\star$ has no effect on the product of $\varphi(\mathrm{x})$ with any $\omega\in V\left(\Phi^e\right)$:

$$\varphi(\mathrm{x})\star\omega=\varphi(\mathrm{x})\omega,\qquad \omega\star\varphi(\mathrm{x})=\omega\varphi(\mathrm{x}),\qquad \text{and}\quad \text{herm. conj.} \tag{92}$$

(as usual, these products are well defined only if ω is x-independent). Consequently, for any $\mathrm{x},\mathrm{y}\in\{\mathrm{x}_1,\mathrm{x}_2,\ldots\}$ the CCR (89) can be rewritten in the form

$$[\varphi(\mathrm{x})\stackrel{\star}{,}\varphi(\mathrm{y})]_\mp=\text{h.c.}=0,\qquad [\varphi(\mathrm{x})\stackrel{\star}{,}\varphi^{*_\star}(\mathrm{y})]_\mp=\varphi_i(\mathrm{x})\star\varphi_i^{*_\star}(\mathrm{y}) \tag{93}$$

(here and below $[A\stackrel{\star}{,}B]_\mp:=A\star B\mp B\star A$). By lemma 1 in the appendix, in $V(\Phi^e_\star)=V(\Phi^e)[[\lambda]]$ one can re-express the field itself in terms of $\star$-products:

$$\varphi(\mathrm{x})=\varphi_i(\mathrm{x})\star a'^i,\qquad \varphi^*(\mathrm{x})=\varphi^{*_\star}(\mathrm{x})=a_i^+\star\varphi_i^{*_\star}(\mathrm{x}). \tag{94}$$

The $\star$-commutation rules among $a'^i, a_j^+, \varphi_i(\mathrm{x}), \varphi_i(\mathrm{y}), \varphi_i^{*_\star}(\mathrm{x}), \varphi_i^{\hat{*}}(\mathrm{y})$ besides (71) are reported in formula (A.8) in the appendix. In particular, a'^i, a_j^+ do not $\star$-commute with functions of x (for all $\mathrm{x}\in\{\mathrm{x}_1,\mathrm{x}_2,\ldots.\}$). If $\mathcal{F}$ is such that one can define the map $\wedge$ on $\mathcal{X}$, in the 'hat notation' (94), (93) become the first two lines in (107). The undeformed fields are characterized by the 'kinematical' property that setting

$$\begin{aligned}\left[\pi^n_\pm(s)\right](\mathrm{x}_1,\ldots,\mathrm{x}_n)&:=\frac{1}{\sqrt{n!}}\left\langle\varphi^*(\mathrm{x}_1)\cdots\varphi^*(\mathrm{x}_n)\Psi_0,s\right\rangle,\\ \Pi^n_\pm(\psi)&:=\frac{1}{\sqrt{n!}}\int_X\mathrm{d}\nu(\mathrm{x}_1)\cdots\int_X\mathrm{d}\nu(\mathrm{x}_n)\varphi^*(\mathrm{x}_1)\cdots\varphi^*(\mathrm{x}_n)\Psi_0\,\psi(\mathrm{x}_1,\ldots,\mathrm{x}_n)\end{aligned} \tag{95}$$

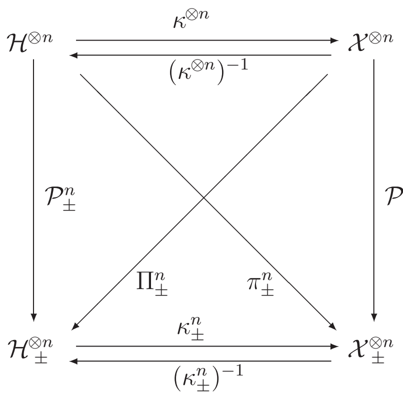

Figure 1. Commutative diagram.

(the scalar products in the first, second line are in $\mathcal{H}^{\otimes n}$, $\mathcal{X}^{\otimes n}$ respectively) defines 'crossed' (anti)symmetric projectors, i.e. linear maps $\pi^n_\pm : \mathcal{H}^{\otimes n} \mapsto \mathcal{X}^{\otimes n}_\pm$ and $\Pi^n_\pm : \mathcal{X}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}_\pm$ making diagram 1 commutative (see the appendix). If $\psi = f_1 \otimes \cdots \otimes f_n$ one finds

$$\Pi^n_\pm(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n!}}\varphi^*(f_1)\cdots\varphi^*(f_n)\Psi_0, \qquad \varphi^*(f) := \int_X \mathrm{d}\nu(\mathrm{x})\varphi^*(\mathrm{x})f(\mathrm{x});$$

since for $f_l = \varphi_{i_l}$ the lhs is proportional to $e^\pm_{i_1,\ldots,i_n} \in \mathcal{B}^n_\pm$, the well-known statement that polynomials in $\varphi^*(f)$ (fields smeared with test functions f) applied to the vacuum make up a subspace dense in the Fock space follows. The restrictions of $\pi^n_\pm$, $\Pi^n_\pm$ to $\mathcal{H}^{\otimes n}_\pm$, $\mathcal{X}^{\otimes n}_\pm$ respectively reduce to $\kappa^n_\pm$, $\left(\kappa^n_\pm\right)^{-1}$, in particular give κ, κ^{-1} for $n = 1$.

$\star$-Deformation preserves these properties. After linearly extending $\pi^n_\pm$, $\Pi^n_\pm$ resp. to $\mathcal{H}^{\otimes n}[[\lambda]]$, $\mathcal{X}^{\otimes n}[[\lambda]]$, by (92), (94) the rhs(95)'s do not change as elements resp. of $V\big(\mathcal{X}^{\otimes n}_\pm\big)[[\lambda]] = V\left[\big(\mathcal{X}^{\otimes n}_\pm\big)_\star\right]$ and $\mathcal{H}^{\otimes n}_\pm[[\lambda]]$ if we replace all products by $\star$-products and $*$'s by $*_\star$'s. Hence, if $\wedge : \mathcal{X}[[\lambda]] \mapsto \hat{\mathcal{X}}$ exists, $\hat{\pi}^n_\pm := \wedge^n \circ \pi^n_\pm$, $\hat{\Pi}^n_\pm := \Pi^n_\pm \circ (\wedge^n)^{-1}$ fulfil

$$\begin{aligned}
&\hat{\pi}^n_\pm : \mathcal{H}^{\otimes n}[[\lambda]] \mapsto \widehat{\mathcal{X}^{\otimes n}_\pm} \qquad \left[\hat{\pi}^n_\pm(s)\right](\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n) = \frac{1}{\sqrt{n!}}\left\langle \hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}}_1)\cdots\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}}_n)\Psi_0, s\right\rangle,\\
&\hat{\Pi}^n_\pm : \widehat{\mathcal{X}^{\otimes n}} \mapsto \mathcal{H}^{\otimes n}_\pm[[\lambda]]\\
&\hat{\Pi}^n_\pm(\hat{\psi}) = \frac{1}{\sqrt{n!}}\int_{\hat{X}} \mathrm{d}\hat{\nu}(\hat{\mathrm{x}}_1)\cdots\int_{\hat{X}} \mathrm{d}\hat{\nu}(\hat{\mathrm{x}}_n)\hat{\psi}(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n)\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}}_1)\cdots\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}}_n)\Psi_0
\end{aligned} \tag{96}$$

and analogous properties. Polynomials in $\hat{\varphi}^*(\hat{f}) := \int \mathrm{d}\hat{\nu}\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})\hat{f}(\hat{\mathrm{x}})$ applied to the vacuum make up a subspace dense in the Fock space. $s = e_{i_1} \otimes \cdots \otimes e_{i_n}$ is mapped into

$$\pi^n_\pm(s)(\mathrm{x}_1,\ldots,\mathrm{x}_n) = \varphi_{(i_1}(\mathrm{x}_1)\ldots\varphi_{i_n]}(\mathrm{x}_n), \tag{97}$$

$$\hat{\pi}^n_\pm(s)(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n) = F^{n\,j_1\ldots j_n}_{\;\;(i_1\ldots i_n]}\hat{\varphi}_{j_1}(\hat{\mathrm{x}}_1)\ldots\hat{\varphi}_{j_n}(\hat{\mathrm{x}}_n), \tag{98}$$

where $(\ldots]$ means (anti)symmetrization of the indices. The same result is obtained if $Ns = e^\pm_{i_1,\ldots,i_n} \in \mathcal{B}^n_\pm$. If $Ns = e'^\pm_{i_1,\ldots,i_n} \in \mathcal{B}'^n_\pm$ (as defined after (83)) we find instead

$$\left[\hat{\pi}^n_\pm(s)\right](\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n) = \mathcal{P}^{n,F\,j_1\ldots j_n}_{\pm\;\;i_1\ldots i_n}\hat{\varphi}_{j_1}(\hat{\mathrm{x}}_1)\cdots\hat{\varphi}_{j_n}(\hat{\mathrm{x}}_n) = \mathcal{P}^{n\,h_1\ldots h_n}_{\pm 1\ldots n}\hat{\varphi}_{i_1}(\hat{\mathrm{x}}_{h_1})\cdots\hat{\varphi}_{i_n}(\hat{\mathrm{x}}_{h_n}) \tag{99}$$

($h_l \in \{1,\ldots,n\}$, whereas $i_l, j_l \in \mathbb{N}$), see the appendix. In particular, if $n = 1, 2$, $i_1 \neq i_2$

$$[\hat{\kappa}(e_i)](\hat{\mathrm{x}}) \equiv [\hat{\pi}(e_i)](\hat{\mathrm{x}}) = \left\langle\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})\Psi_0, e_i\right\rangle = \left\langle\Psi_0, \hat{\varphi}(\hat{\mathrm{x}})\hat{a}^+_i\Psi_0\right\rangle = \hat{\varphi}_i(\hat{\mathrm{x}}), \tag{100}$$

$$\big[\hat\kappa^2_\pm\big(\tfrac{1}{\sqrt{2}}e'^{\pm}_{i_1i_2}\big)\big](\hat{\mathrm{x}},\hat{\mathrm{y}}) \equiv \big[\hat\pi^2_\pm\big(\tfrac{1}{\sqrt{2}}e'^{\pm}_{i_1i_2}\big)\big](\hat{\mathrm{x}},\hat{\mathrm{y}})$$
$$= \tfrac{1}{2}\big\langle\Psi_0, \hat\varphi(\hat{\mathrm{y}})\hat\varphi(\hat{\mathrm{x}})\hat a^+_{i_1}\hat a^+_{i_2}\Psi_0\big\rangle = \mathcal{P}^{2,F\,j_1j_2}_{\pm\ \ i_1i_2}\hat\varphi_{j_1}(\hat{\mathrm{x}})\hat\varphi_{j_2}(\hat{\mathrm{y}})$$
$$= \tfrac{1}{2}[\hat\varphi_{i_1}(\hat{\mathrm{x}})\hat\varphi_{i_2}(\hat{\mathrm{y}}) \pm R^{j_2j_1}_{i_1i_2}\hat\varphi_h(\hat{\mathrm{x}})\hat\varphi_{j_2}(\hat{\mathrm{y}})] = \tfrac{1}{2}[\hat\varphi_{i_1}(\hat{\mathrm{x}})\hat\varphi_{i_2}(\hat{\mathrm{y}}) \pm \hat\varphi_{i_1}(\hat{\mathrm{y}})\hat\varphi_{i_2}(\hat{\mathrm{x}})]. \qquad (101)$$

4.3. Field equations of motion and Heisenberg picture

Assume that the n-particle wavefunction $\psi^{(n)}$ fulfils the Schrödinger equation (2) if $n = 1$, and

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi^{(n)} = \mathsf{H}^{(n)}_\star\psi^{(n)}, \qquad \mathsf{H}^{(n)}_\star := \sum_{h=1}^{n}\mathsf{H}^{(1)}_\star(\mathrm{x}_h, \partial_{\mathrm{x}_h}, t) + \sum_{h<k}W(\rho_{hk})\star \qquad (102)$$

if $n \geqslant 2$; here the time coordinate t remains 'commuting'. The class is sufficiently general to include $\star$-deformations of the Schrödinger equations of many physically relevant models: it includes possible $\star$-local interactions with the external background potential $V(\mathrm{x}, t)$ and $U(1)$ gauge potential $\mathbf{A}(\mathrm{x}, t)$ (q ≡electrical charge of the particle) as well as internal interactions through a two-body potential W depending only on the (invariant) distance $\rho_{hk} = |\mathrm{x}_h - \mathrm{x}_k|$ between $\mathrm{x}_h, \mathrm{x}_k$. $\mathsf{H}^{(n)}_\star$ will be Hermitian provided $\mathsf{H}^{(1)}$ is and $\beta \triangleright \mathsf{H}^{(1)} = \mathsf{H}^{(1)}$, as we shall assume. In general (102) is a $\star$-differential, pseudodifferential equation, preserving the (anti)symmetry of $\psi^{(n)}$. The Fock space Hamiltonian

$$\mathsf{H}_\star(\varphi) = \int_X \mathrm{d}\nu(\mathrm{x})\varphi^{\hat *}(\mathrm{x}) \star \mathsf{H}^{(1)}_\star\varphi(\mathrm{x}) \star$$
$$+ \int_X \mathrm{d}\nu(\mathrm{x})\int_X \mathrm{d}\nu(\mathrm{y})\varphi^{\hat *}(\mathrm{y}) \star \varphi^{\hat *}(\mathrm{x}) \star W(\rho_{\mathrm{xy}}) \star \varphi(\mathrm{x}) \star \varphi(\mathrm{y}) \star$$

annihilates the vacuum, commutes with the number-of-particles operator $\mathbf{n} := a^+_i \star a^i$ and its restriction to $\mathcal{H}^{\otimes n}_\pm$ coincides with $\mathsf{H}^{(n)}_\star$ up to the unitary transformation $\tilde\kappa^{\otimes n}$ (this can be easily checked using (95)). As in the undeformed case, in the Fock space one can also consider Hamiltonians not commuting with $\mathbf{n}$; consequently, the latter will no longer be a constant of motion.

We now introduce the evolution operator $U(t) := T\big[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t\,\mathsf{H}_\star}\big]$ and go from the Schrödinger to the *Heisenberg picture*. The Heisenberg field $\varphi^H_\star(\mathrm{x}, t) := [U(t)]^{*_\star}\varphi(\mathrm{x})U(t)$ fulfils the equal-time commutation relations

$$[\varphi^H_\star(\mathrm{x}, t) \stackrel{\star}{,} \varphi^H_\star(\mathrm{y}, t)]_\mp = \text{h.c.} = 0, \qquad \big[\varphi^H_\star(\mathrm{x}, t) \stackrel{\star}{,} \varphi^{H*_\star}_\star(\mathrm{y}, t)\big]_\mp = \varphi_i(\mathrm{x}) \star \varphi^{*_\star}_i(\mathrm{y}) \qquad (103)$$

and the evolution equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi^H_\star = \big[\varphi^H_\star, \mathsf{H}_\star\big]. \qquad (104)$$

If $\mathsf{H}^{(1)}_\star$ is t-independent, so is $\mathsf{H}_\star$, then $\mathsf{H}_\star(\varphi^H_\star) = \mathsf{H}_\star(\varphi)$, and (103) and (104) can be formulated directly in the Heisenberg picture as equations in the unknown $\varphi^H_\star(t)$. If $W = 0$ (104) formally coincides with (2),

$$\mathrm{i}\hbar\frac{\partial\varphi^H_\star}{\partial t} = \mathsf{H}^{(1)}_\star\varphi^H_\star, \qquad (105)$$

a $\star$-differential equation; following the conventional terminology we shall call 'second quantization' the replacement of the wavefunction ψ by the field operator $\varphi^H_\star$. If in addition $\mathsf{H}^{(1)}_\star$ is t-independent (in particular, in the free case $V \equiv 0 \equiv A^a$), setting $x := (t, \mathrm{x})$ we can express the Heisenberg field solution of (105) in the form

$$\varphi^H_\star(x) = \varphi_i(x) \star a'^i \quad (\text{sum over } i), \qquad [\kappa^H(t)](e_i) = \varphi_i(\mathrm{x})\mathrm{e}^{-\mathrm{i}\omega_i t} =: \varphi_i(x) \qquad (106)$$

in terms of an orthormal basis $\{\varphi_i(\mathrm{x})\}$ of eigenfunctions of $\mathsf{H}^{(1)}_\star$ with eigenvalues $\hbar\omega_i$ (the sum over i actually becomes an integral over the continuous part of the spectrum). Now we assume $\mathcal{F}$ such that $\widehat{\mathcal{X}}, \widehat{\mathcal{D}}, \wedge : \mathcal{X} \mapsto \widehat{\mathcal{X}}$ exist and $V, A^a, W \in \mathcal{X}$, so that $\hat{V} = \wedge(V)$, $\hat{A}^a = \wedge(A^a)$, $\hat{W} = \wedge(W)$ are well defined (actually, $\hat{W} = W$ by the $U\mathbf{g}$-invariance of ρ_{hk}). The map $\hat{\kappa}^{n,H}_\pm(t) = \hat{\kappa}^n_\pm \circ U(t) : \mathcal{H}^{\otimes n}_\pm \mapsto \widehat{\mathcal{X}^{\otimes n}_\pm}$ is a t-dependent $\widehat{U\mathbf{g}}$-equivariant noncommutative configuration space realization of $\mathcal{H}^{\otimes n}_\pm$ (for $n = 1$ we denote it by $\hat{\kappa}^H$). We shall denote $\hat{\kappa}^H(e_i) = \hat{\varphi}_i(\hat{x})$. The analogues of (96) are now

$$\hat{\psi}_s(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n,t) := \left\{\hat{\pi}^n_\pm[U(t)s]\right\}(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n) = \frac{1}{\sqrt{n!}}\left\langle\hat{\varphi}^{H\hat{*}}(\hat{\mathrm{x}}_n,t)\cdots\hat{\varphi}^{H\hat{*}}(\hat{\mathrm{x}}_1,t)\Psi_0,s\right\rangle,$$

$$s = \hat{\Pi}^n_\pm[\hat{\psi}_s(t)] := \frac{1}{\sqrt{n!}}\int_{\hat{X}} \mathrm{d}\hat{\nu}(\hat{\mathrm{x}}_1)\cdots\int_{\hat{X}}\mathrm{d}\hat{\nu}(\hat{\mathrm{x}}_n)\hat{\varphi}^{H\hat{*}}(\hat{\mathrm{x}}_1,t)\cdots\hat{\varphi}^{H\hat{*}}(\hat{\mathrm{x}}_n,t)\Psi_0\hat{\psi}_s(\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_n,t);$$

$\hat{\psi}_s$ fulfils the Schrödinger equation $(107)_4$ and $\hat{\psi}_s(t=0) = \hat{\pi}^n_\pm(s)$. Replacing $\hat{V}, \hat{\mathbf{A}}, \hat{\varphi}_i, \hat{\psi}^{(n)}, \int_{\hat{X}}\mathrm{d}\hat{\nu}(\hat{\mathrm{x}})$ in the previous equations we can reformulate the latter using only 'hatted' objects:

$$\hat{\varphi}(\hat{\mathrm{x}}) = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{a}'^i, \qquad \varphi^{\hat{*}}(\hat{\mathrm{x}}) = \hat{a}^+_i\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{x}}),$$

$$[\hat{\varphi}(\hat{\mathrm{x}}),\hat{\varphi}(\hat{\mathrm{y}})]_\mp = \text{h.c.} = 0, \qquad [\hat{\varphi}(\hat{\mathrm{x}}),\hat{\varphi}^{\hat{*}}(\hat{\mathrm{y}})]_\mp = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{y}}),$$

$$\hat{\mathsf{H}}^{(1)} = \frac{-\hbar^2}{2m}\hat{D}^a\hat{D}_a + \hat{V}, \qquad \hat{D}_a = \hat{\partial}_a + \mathrm{i}q\hat{A}_a, \qquad \hat{\mathsf{H}}^{(n)} = \sum_{h=1}^{n}\hat{\mathsf{H}}^{(1)}(\hat{\mathrm{x}}_h,\hat{\partial}_{\mathrm{x}_h},t) + \sum_{h<k}\hat{W}(\hat{\rho}_{hk}),$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\psi}^{(n)} = \hat{\mathsf{H}}^{(n)}\hat{\psi}^{(n)},$$

$$\hat{\mathsf{H}} = \int_{\hat{X}}\mathrm{d}\hat{\nu}(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})\hat{\mathsf{H}}^{(1)}(\hat{\mathrm{x}},\hat{\partial}_{\mathrm{x}},t)\hat{\varphi}(\hat{\mathrm{x}}) + \int_{\hat{X}}\mathrm{d}\hat{\nu}(\hat{\mathrm{x}})\int_{\hat{X}}\mathrm{d}\hat{\nu}(\hat{\mathrm{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})W(\hat{\rho}_{\mathrm{xy}})\hat{\varphi}(\mathrm{x})\hat{\varphi}(\mathrm{y}),$$

$$[\hat{\varphi}^H(\hat{\mathrm{x}},t),\hat{\varphi}^H(\hat{\mathrm{y}},t)]_\mp = \text{h.c.} = 0, \qquad [\hat{\varphi}^H(\hat{\mathrm{x}},t),\hat{\varphi}^{H\hat{*}}(\hat{\mathrm{y}},t)]_\mp = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{y}}),$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\varphi}_H = [\hat{\varphi}^H,\hat{\mathsf{H}}],$$

$$\hat{\varphi}^H(\hat{x}) = \hat{\varphi}_i(\hat{x})\hat{a}'^i, \qquad \varphi^{\hat{*}}_H(\hat{x}) = \hat{a}^+_i\hat{\varphi}^{\hat{*}}_i(\hat{x}) \qquad \text{if } \hat{W} = 0 \text{ and } \partial_t\mathsf{H}^{(1)} = 0 \tag{107}$$

($\hat{x},\hat{y} \in \{\hat{x}_1,\hat{x}_2,\ldots\}$ are two sets of noncommutative coordinates fulfilling (43)). Summing up, at least formally, *we can formulate the same quantum theory on either the commutative or the noncommutative space*: (43), (50) and (107) summarize a candidate framework for non-relativistic *field quantization on the noncommutative spacetime* $\mathbb{R}\times\hat{X}$ *compatible with QM axioms and Bose–Fermi statistics*, in that it has been obtained by a *second quantization* procedure. Note that the field commutation relations, in both the Schrödinger and Heisenberg picture, are of the type 'field (anti)commutator = a distribution'. The framework is not only $\widehat{U\mathbf{g}}$-, but also $\widehat{U\mathbf{g}'}$-covariant ($\mathbf{g}'$ is the Galilei Lie algebra); to account for the t-dependence $C^1(\mathbb{R},\mathcal{H}), C^1(\mathbb{R},\mathcal{X}),\ldots$ must replace $\mathcal{H},\mathcal{X},\ldots$ as carrier spaces of the representations. Now one can forget how we have got it and investigate case by case its consistency beyond the level of formal λ-power series, using only 'noncommutative mathematics'.

Remark 1. In general (102) are pseudodifferential (and therefore highly non-local) equations, but of second order as $\star$-differential equations, and similarly (104). However, by (15) the Hamiltonians $\mathsf{H}^{(1)}_\star, \mathsf{H}^{(n)}_\star, \mathsf{H}_\star$ coincide with the local $\mathsf{H}^{(1)}, \mathsf{H}^{(n)}, \mathsf{H}$ (defined by (102) *without*

$\star$-products) if $V, \mathbf{A}$ are H_s-invariant, where H_s is the smallest Hopf $*$-subalgebra $H_s \subseteq U\mathbf{g}$ such that $\mathcal{F} \in (H_s \otimes H_s)[[\lambda]]$ (the Laplacian $\Delta = |g|^{-\frac{1}{2}}\partial^a\partial_a$ and $W(\rho)$ are already $U\mathbf{g}$-invariant); then the dynamics remains undeformed. If so there is no simplification in treating the dynamics in the noncommutative setting, although this is possible. This may become convenient if $\hat{V} \neq V$ or $\hat{\mathbf{A}} \neq \mathbf{A}$ and the dependence of $\hat{V}, \hat{\mathbf{A}}$ on $\hat{\mathrm{x}}$ is simpler than the dependence of $V, \mathbf{A}$ on x, e.g. are solutions themselves of $\star$-differential equations which are truly pseudodifferential. See sections 5 and 3 in [32] for examples.

5. Non-relativistic QM on Moyal–Euclidean space

$X = \mathbb{R}^3$, $G = ISO(3)$, the twist is 51 and we use the results of section 3.3.1. It is instructive to see explicitly how $\wedge^n$, acting on (anti)symmetric wavefunctions, 'hides' their (anti)symmetry. Sticking to $n = 2$, we find e.g. on the generalized basis of (anti)symmetrized plane waves

$$\wedge^2(\mathrm{e}^{\mathrm{i}q_1\cdot\mathrm{x}_1}\,\mathrm{e}^{\mathrm{i}q_2\cdot\mathrm{x}_2} \pm \mathrm{e}^{\mathrm{i}q_2\cdot\mathrm{x}_1}\,\mathrm{e}^{\mathrm{i}q_1\cdot\mathrm{x}_2}) = \mathrm{e}^{\mathrm{i}q_1\cdot\hat{\mathrm{x}}_1}\,\mathrm{e}^{\mathrm{i}q_2\cdot\hat{\mathrm{x}}_2}e^{\frac{\mathrm{i}}{2}q_1\theta q_2} \pm \mathrm{e}^{\mathrm{i}q_2\cdot\hat{\mathrm{x}}_1}\,\mathrm{e}^{\mathrm{i}q_1\cdot\hat{\mathrm{x}}_2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}q_1\theta q_2}.$$

As noted in section 3 of [32], the (anti)symmetry remains manifest if we use coordinates ξ_i^a, X^a of type (63) with $X^a = \sum_{i=1}^n \mathrm{x}_i^a/n$ being the coordinates of the centre-of-mass of the system (which are completely symmetric). By (65), the map $\wedge^n$ deforms only the X part of the wavefunction, leaving unchanged and completely (anti)symmetric the ξ-part. For instance, the previous equation becomes

$$\wedge^2[\mathrm{e}^{\mathrm{i}(q_1+q_2)\cdot X}(\mathrm{e}^{\mathrm{i}(q_2-q_1)\cdot\xi_1} \pm \mathrm{e}^{-\mathrm{i}(q_2-q_1)\cdot\xi_1})] = \mathrm{e}^{\mathrm{i}(q_1+q_2)\cdot\hat{X}}(\mathrm{e}^{\mathrm{i}(q_2-q_1)\cdot\hat{\xi}_1} \pm \mathrm{e}^{-\mathrm{i}(q_2-q_1)\cdot\hat{\xi}_1}).$$

By remark 1, if $V, \mathbf{A}$ are translation invariant (i.e. constant, in particular vanish), the non-locality of the Hamiltonians disappears[21], and the dynamics reduces to the undeformed one. Then formulating the dynamics in the $\star$-deformed, noncommutative setting brings no formal advantage in solving the equation of motion. Therefore, we consider two very simple choices of non-constant vector potential (and $A^0 \equiv V = 0$). They fulfil the Coulomb gauge condition and the free field equation not only in the standard differential version $A^0 = \partial_i A^i = 0$, $\partial_\mu F^{\mu i} = \Box A^i = 0$, $F^{\mu\nu} := \partial^{[\mu}A^{\nu]}$, but also in the $\star$-differential version

$$A^0 = \partial_i \star A^i = 0, \quad \partial_\mu F_\star^{\mu i} + \mathrm{i}e[A_\mu \stackrel{\star}{,} F_\star^{\mu i}] = 0, \quad F_\star^{\mu\nu} := \partial^{[\mu}A^{\nu]} - \frac{e^2}{2}[A^\mu \stackrel{\star}{,} A^\nu]. \tag{108}$$

1. Charged particle in a constant magnetic field $\mathbf{B}$. The simplest gauge choice is $A^i(\mathrm{x}) = \epsilon^{ijk}B^j\mathrm{x}^k/2$. One finds

$$\mathsf{H}_\star^{(1)} = -\frac{\hbar^2}{2m}\left\{\Delta - \frac{\mathrm{i}q}{\hbar c}\varepsilon^{abc}B^a\mathrm{x}^b\partial^c - \frac{q^2}{4\hbar^2c^2}\left[\mathbf{B}^2\mathrm{x}^2 - (\mathbf{B}\cdot\mathrm{x})^2\right] - \frac{q}{2\hbar c}\varepsilon^{abc}B^a\partial^b\theta^{cd}\partial^d \right.$$
$$\left. - \frac{q^2}{16\hbar^2c^2}[\mathbf{B}^2[4\mathrm{i}(\mathrm{x}\theta\partial) + (\theta^2)^{ab}\partial^a\partial^b] - 4\mathrm{i}(\mathbf{B}\cdot\mathrm{x})(\mathrm{x}\theta\partial) + 4(\mathbf{B}\theta\partial)^2]\right\}$$

(we have displayed the undeformed Hamiltonian in the first line and the corrections in the second line), so in $\mathcal{X}$ the Schrödinger equation is still differential of second order, but more complicated than in the commutative case. We show that in terms of 'hatted' objects it can be formulated and solved as in the undeformed case. We choose the x^3-axis so that $q\mathbf{B} = qB\vec{k}$ with $qB > 0$; this gives

$$\hat{D}^3 = \hat{\partial}^3, \quad \hat{D}^1 = \hat{\partial}^1 - \mathrm{i}\beta\hat{\mathrm{x}}^2, \quad \hat{D}^2 = \hat{\partial}^2 + \mathrm{i}\beta\hat{\mathrm{x}}^1 \qquad \Rightarrow$$
$$[\hat{\partial}^3, \hat{D}^a] = 0, \quad [\hat{D}^1, \hat{D}^2] = \mathrm{i}2\beta\left[1 - \frac{\beta\theta^{12}}{2}\right],$$

[21] Actually, to this end it is sufficient that $V, \mathbf{A}$ are constant along each plane perpendicular to the vector of components $\theta^a := \varepsilon^{abc}\theta^{bc}/2$.

where $a = 1, 2$ and $\beta := qB/2\hbar c$. We assume $qB\theta^{12} < 4\hbar c$. Then one finds

$$a := \alpha[\hat{D}^1 - \mathrm{i}\hat{D}^2], \;\; \alpha := \left[4\beta\left(1 - \frac{\beta\theta^{12}}{2}\right)\right]^{\frac{-1}{2}} \quad \Rightarrow \quad a^* = -\alpha[\hat{D}^1 + \mathrm{i}\hat{D}^2], \;\; [a, a^*] = 1, \tag{109}$$

$$\begin{aligned} \hat{\mathsf{H}}^{(1)} &= \frac{-\hbar^2}{2m}\hat{D}^i\hat{D}^i = \frac{-\hbar^2}{2m}\left[(\hat{\partial}^3)^2 - \frac{1}{2\alpha^2}(aa^* + a^*a)\right] = \hat{\mathsf{H}}^{(1)}{}_\parallel + \hat{\mathsf{H}}^{(1)}{}_\perp, \\ \hat{\mathsf{H}}^{(1)}{}_\parallel &:= \frac{(-i\hbar\hat{\partial}^3)^2}{2m}, \qquad \hat{\mathsf{H}}^{(1)}{}_\perp := \hbar\omega\left(a^*a + \frac{1}{2}\right), \qquad \omega := \frac{qB}{mc}\left(1 - \frac{\beta\theta^{12}}{2}\right). \end{aligned} \tag{110}$$

As $\hat{\partial}^3$ commutes with a, a^*, the operators $\hat{\mathsf{H}}^{(1)}{}_\parallel, \hat{\mathsf{H}}^{(1)}{}_\perp$ commute with each other. The first is as on the commutative space, and has the continuous spectrum $[0, \infty[$; the generalized eigenfuntions are the eigenfuntions $\mathrm{e}^{\mathrm{i}k\hat{x}^3}$ of $p^3 = -\mathrm{i}\hbar\hat{\partial}^3$ with the eigenvalue $\hbar k$. The second is a harmonic oscillator Hamiltonian with ω being modified by the noncommutativity through θ^{12} (but not θ^{13}, θ^{23}). So the spectrum of $\hat{\mathsf{H}}^{(1)}$ is the set of $E_{n,k^3} = \hbar\omega(n + 1/2) + (\hbar k^3)^2/2m$.

To find a basis of eigenfunctions we define in analogy with the undeformed case

$$\hat{z} := \sqrt{\frac{\beta}{2}}(\hat{x}^1 + \mathrm{i}\hat{x}^2), \quad \partial_{\hat{z}} := \frac{1}{\sqrt{2\beta}}(\hat{\partial}_1 - \mathrm{i}\hat{\partial}_2) \quad \Rightarrow \quad \hat{z}^* = \sqrt{\frac{\beta}{2}}(\hat{x}^1 - \mathrm{i}\hat{x}^2), \quad \partial_{\hat{z}}^* = -\partial_{\hat{z}^*}$$

and find that the only nontrivial commutators among $\hat{z}, \hat{z}^*, \partial_{\hat{z}}, \partial_{\hat{z}^*}$ are

$$[\partial_{\hat{z}}, \hat{z}] = 1, \qquad [\partial_{\hat{z}^*}, \hat{z}^*] = 1, \qquad [\hat{z}, \hat{z}^*] = \beta\theta^{12}.$$

We can thus re-express a, a^* in the form

$$a = \alpha\sqrt{2\beta}(\hat{z}^* + \partial_{\hat{z}}), \qquad a^* = \alpha\sqrt{2\beta}(\hat{z} - \partial_{\hat{z}^*}).$$

Setting $\hat{l}^3 := \hat{z}\partial_{\hat{z}} - \hat{z}^*\partial_{\hat{z}^*} - \beta\theta^{12}\partial_{\hat{z}}\partial_{\hat{z}^*}$ and $\mathrm{n} := a^*a$ we also find

$$\begin{aligned} &[\hat{l}^3, \hat{z}^*] = -\hat{z}^*, \qquad [\hat{l}^3, \hat{z}] = \hat{z}, \qquad [\hat{l}^3, a^*] = a^*, \\ &[\hat{l}^3, a] = -a, \qquad [\hat{l}^3, \mathrm{n}] = 0. \end{aligned} \tag{111}$$

The existence of 'ground-state' eigenfunctions $\hat{\psi}_0$ characterized by the condition $a\hat{\psi}_0 = 0$ is proved as in the commutative case. As $[a, \hat{z}^*] = 0$, if $\hat{\psi}_0$ fulfils this condition, so does $(\hat{z}^*)^r\hat{\psi}_0$ for all $r \in \mathbb{N}$; moreover, if $\hat{l}^3\hat{\psi}_0 = m\hat{\psi}_0$, then $\hat{l}^3(\hat{z}^*)^r\hat{\psi}_0 = (m - r)(\hat{z}^*)^r\hat{\psi}_0$. In particular we find

$$\hat{\psi}_{0,0}(\hat{z}^*, \hat{z}) := \int \mathrm{d}k\, \mathrm{d}k^* \mathrm{e}^{\mathrm{i}k\hat{z}^*}\, \mathrm{e}^{\mathrm{i}k^*\hat{z}}\, \mathrm{e}^{-kk^*} \quad \Rightarrow \quad a\hat{\psi}_{0,0} = 0 = \hat{l}^3\hat{\psi}_{0,0}. \tag{112}$$

In analogy with the undeformed case we choose as a complete set of commuting observables $\{p^3, \mathrm{n}, \hat{l}^3\}$. The deformed Landau eigenfunctions

$$\hat{\psi}_{k^3,n,m}(\hat{x}) = (a^*)^n(\hat{z}^*)^{n-m}\hat{\psi}_{0,0}(\hat{z}^*, \hat{z})\, \mathrm{e}^{\mathrm{i}k^3\hat{x}^3} \tag{113}$$

are generalized eigenfunctions with eigenvalues $p^3 = \hbar k^3 \in \mathbb{R}$, $\mathrm{n} = n = 0, 1, \ldots$, $\hat{l}^3 = m = n, n-1, \ldots$ and build up an orthogonal basis of $\mathcal{L}^2(\mathbb{R}^3)$. They are also eigenfunctions of $\hat{\mathsf{H}}^{(1)}$ with eigenvalues $E_{n,k^3} = \hbar\omega(n+1/2)+(\hbar k^3)^2/2m$. Replacing $\hat{x}^a \to x^a\star$ and performing all the $\star$-products one finds the corresponding eigenfuntions $\psi_{k^3,n,m}(\mathrm{x})$ of $\mathsf{H}^{(1)}_\star$. To constrain the size of this paper we do not consider multi-particle systems here.

2. *Charged particle in a plane wave electromagnetic field.* $A^a(\mathrm{x}) = \varepsilon^a(\mathrm{p})\exp[-\mathrm{i}p \cdot x] \equiv \varepsilon^a(\mathrm{p})\exp[\mathrm{i}(\mathrm{p}\cdot\mathrm{x} - |\mathrm{p}|t)]$ (the amplitude vector fulfilling $\varepsilon^a(\mathrm{p})\mathrm{p}^a = 0$). To check (108) it is useful to use the properties

$$\mathrm{e}^{\mathrm{i}\mathrm{p}\cdot\mathrm{x}} \star f(\mathrm{x}) = \mathrm{e}^{\mathrm{i}\mathrm{p}\cdot\mathrm{x}} f(\mathrm{x} + \theta\mathrm{p}/2) \quad \Rightarrow \quad \mathrm{e}^{\mathrm{i}\mathrm{p}\cdot\mathrm{x}} \star \mathrm{e}^{\mathrm{i}a\mathrm{p}\cdot\mathrm{x}} = \mathrm{e}^{\mathrm{i}\mathrm{p}\cdot\mathrm{x}}\, \mathrm{e}^{\mathrm{i}a\mathrm{p}\cdot\mathrm{x}}$$

where $(\theta \mathrm{p})^a := \theta^{ab}\mathrm{p}^b$, as $\mathrm{p}\theta\mathrm{p} = 0$. The Schrödinger equation (102) for $n = 1$ particle becomes

$$i\hbar\partial_t\psi(\mathrm{x},t) = \frac{-\hbar^2}{2m}\left[\Delta\psi(\mathrm{x},t) + 2iq\,\mathrm{e}^{-ip\cdot x}\varepsilon^a\hat{\partial}_a\psi\left(\mathrm{x}+\frac{\theta\mathrm{p}}{2},t\right) - q^2\,\mathrm{e}^{-2ip\cdot x}|\varepsilon|^2\psi(\mathrm{x}+\theta\mathrm{p},t)\right];$$

the non-locality induced by the $\star$-product is here particularly simple, in that it involves the wavefunction at points $\mathrm{x}, \mathrm{x}+\theta\mathrm{p}/2, \mathrm{x}+\theta\mathrm{p}$ related by the constant shift $\theta\mathrm{p}/2$. As in the undeformed case, one can approximate the dynamics by perturbative methods.

6. Relativistic second quantization

Now we adopt as a starting point Minkowski spacetime (i.e. $\mathbb{R}^4$ endowed with Minkowski metric) as a pseudo-Riemannian manifold X; we shall denote as x^μ ($\mu = 0,1,2,3$) the coordinates w.r.t. a fixed inertial frame. The isometry group is the Poincaré Lie group whose Lie algebra we denote as $\mathcal{P}$; we denote as $P_\mu, M_{\mu\nu}$ the generators of spacetime translations and Lorentz transformations respectively. A relativistic particle is described by choosing as the algebra of observables $\mathcal{O} = H = U\mathcal{P}$ and as the Hilbert space $\overline{\mathcal{H}}$ the completion of a pre-Hilbert space $\mathcal{H}$ carrying an irreducible $*$-representation of $U\mathcal{P}$ characterized by a nonnegative eigenvalue m^2 of the Casimir $P^\mu P_\mu$ and a nonnegative spectrum for P^0. For simplicity we stick to the case of a scalar particle (i.e. the Pauli–Lubanski Casimir vanishes) of positive mass m. Normalizable states s, in particular the vectors of an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}$, can be decomposed as combinations (integrals) of generalized eigenstates e_p of the operators P_μ with eigenvalues p_μ $[(p^a) \equiv \mathrm{p} \in \mathbb{R}^3$ and $p^0 \equiv \sqrt{\mathrm{p}^2+m^2} > 0]$: $s = \int \mathrm{d}\mu(p)e_\mathrm{p}\tilde{\psi}_s(\mathrm{p})$, where $\mathrm{d}\mu(\mathrm{p}) = \mathrm{d}^3\mathrm{p}/2p^0$ is the Poincaré invariant measure. Fixing the normalization of e_p by setting as usual $(e_\mathrm{p}, e_\mathrm{q}) = 2p^0\delta^3(\mathrm{p}-\mathrm{q})$, the scalar product is expressed as

$$(s,v) = \int \mathrm{d}\mu(\mathrm{p})\tilde{\psi}_s^*(\mathrm{p})\tilde{\psi}_v(\mathrm{p}).$$

The $\mathcal{P}$-action on $\mathcal{H}$ reads

$$P_\mu \triangleright s = \int \mathrm{d}\mu(\mathrm{p})e_\mathrm{p}p_\mu\tilde{\psi}_s(\mathrm{p}), \qquad M_{\mu\nu}\triangleright s = \int \mathrm{d}\mu(\mathrm{p})e_\mathrm{p}i[p_\mu\partial_{p^\nu} - p_\nu\partial_{p^\mu}]\tilde{\psi}_s(\mathrm{p})$$

(where we have to replace $\partial_{p^0} \to 0$, as p^0 is no longer an independent variable). As $\mathcal{H}$ we take the space of vectors s for which $\tilde{\psi}_s \in \mathcal{S}(\mathbb{R}^3)$; clearly s are in the domain of all elements of H. We denote by a_i, a_i^+ the creation and annihilation operators corresponding to e_i and by $a_\mathrm{p}^+, a^\mathrm{p}$ the generalized ones corresponding to e_p;[22] the former fulfil (70), (68), the latter $P_\mu \triangleright a_\mathrm{p}^+ = p_\mu a_\mathrm{p}^+$, $P_\mu \triangleright a^\mathrm{p} = -p_\mu a^\mathrm{p}$ and

$$[a^\mathrm{p}, a^\mathrm{q}] = 0, \qquad \left[a_\mathrm{p}^+, a_\mathrm{q}^+\right] = 0, \qquad \left[a^\mathrm{p}, a_\mathrm{q}^+\right] = 2p^0\delta^3(\mathrm{p}-\mathrm{q}). \tag{114}$$

A t-dependent configuration space realization (Heisenberg picture) is obtained setting $\kappa^H(e_\mathrm{p}) = \mathrm{e}^{-ip\cdot x}$, $\tilde{\kappa}^H(P_\mu) = i\partial_\mu$, $\tilde{\kappa}^H(M_{\mu\nu}) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, and the 'on-shell' condition $(P^\mu P_\mu - m^2)s = 0$ becomes on $\psi_s = \kappa^H(s) = \int \mathrm{d}\mu(p)\,\mathrm{e}^{-ip\cdot x}\tilde{\psi}_s(\mathrm{p})$ the Klein–Gordon equation $(\Box + m^2)\psi_s = 0$. So $\mathcal{X} := \kappa^H(\mathcal{H})$ is the pre-Hilbert space of normalizable, smooth, positive energy solutions of the K–G equation (these functions depend both on space and time). We denote $\kappa^H(e_i) = \varphi_i$ and $\Phi^e = \mathcal{A}^\pm \otimes \left(\bigotimes_{i=1}^\infty \mathcal{X}'\right)$. The Hermitian relativistic free field (in the Heisenberg picture) is $\varphi(x) = \varphi_i(x)a^i + a_i^+\varphi_i^*(x) = \int \mathrm{d}\mu(p)\,[\mathrm{e}^{-ip\cdot x}a^\mathrm{p} + a_\mathrm{p}^+\,\mathrm{e}^{ip\cdot x}]$;

[22] In order to uniquely fix the signs in the definitions of the 'generalized bases' $\mathcal{B}_\pm^n$ and of the creation and annihilation operators one needs to choose a complete ordering within the set of labels $\mathrm{p} \in \mathbb{R}^3$, since no natural one is available: one can declare e.g. $\mathrm{p} > \mathrm{p}'$ if $p_1 > p_1'$, or $p_1 = p_1'$ and $p_2 > p_2'$, or $p_1 = p_1'$ and $p_2 = p_2'$ and $p_3 > p_3'$. As is known, the physical results are independent of this choice.

it is basis-independent, H-invariant and fulfils the Klein–Gordon equation. The 'crossed' (anti)symmetric projectors $\pi^n_\pm, \Pi^n_\pm$ of figure 1 are defined by replacing in (95) the scalar product integral (85) by the one in (121) and φ^* by the creation part of φ.

Fixing a twist $\mathcal{F} \in (U\mathcal{P} \otimes U\mathcal{P})[[\lambda]]$ (examples are shown in the next subsection and in [12]), we can apply the associated deforming procedure to the whole setting. The deformed annihilation and creation operators associated with $\{e_i\}_{i\in\mathbb{N}}$ will fulfil the commutation relations (71). By (15), $\Box\star\omega = \Box\,\omega$ for all $\omega \in \Phi^e$ (as the d'Alembertian is $U\mathcal{P}$-invariant), so the free field equation and its solutions are not deformed. By lemma 1 one can express $\Box$ in the form $\Box = \partial_\mu \star \partial'^\mu$ $(\partial'^\mu := S(\beta) \triangleright \partial^\mu)$, φ in the form

$$\varphi(x) = \varphi_i(x) \star a'^i + a^+_i \star \varphi^{\star}_i(x) = \int \mathrm{d}\mu(p)\big[\mathrm{e}^{-\mathrm{i}p\cdot x} \star a'^{\mathrm{p}} + a^+_{\mathrm{p}} \star \mathrm{e}^{\mathrm{i}p\cdot x}\big] \tag{115}$$

and the free field commutation relation in the form

$$[\varphi(x) \overset{\star}{,} \varphi(y)] = \varphi_i(x) \star \varphi^{*_\star}_i(y) - \varphi_i(y) \star \varphi^{*_\star}_i(x) = \int \mathrm{d}\mu(p)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

for any $x, y \in \{x_1, x_2, \ldots\}$. As is known, this vanishes if $x - y$ is space-like (microcausality). Provided $\mathcal{F}$ allows the definition of $\widehat{\mathcal{X}}, \widehat{\mathcal{D}}$ and $\wedge : \mathcal{D} \mapsto \widehat{\mathcal{D}}$, and going to the 'hat notation' we find for any two sets $\hat{x}, \hat{y} \in \{\hat{x}_1, \hat{x}_2, \ldots\}$ of noncommutative coordinates fulfilling (43)

$$\begin{aligned} &\hat\varphi(\hat x) = \hat\varphi_i(\hat x)\hat a'^i + \hat a^+_i \hat\varphi^{\star}_i(\hat x) = \int \mathrm{d}\mu(p)\big[\wedge(\mathrm{e}^{-\mathrm{i}p\cdot x})\hat a'^{\mathrm{p}} + \hat a^+_{\mathrm{p}} \wedge (\mathrm{e}^{\mathrm{i}p\cdot x})\big],\\ &[\hat\varphi(\hat x), \hat\varphi(\hat y)] = \hat\varphi_i(\hat x)\hat\varphi^{\star}_i(\hat y) - \hat\varphi_i(\hat y)\hat\varphi^{\star}_i(\hat x),\\ &(\hat\Box + m^2)\hat\varphi(\hat x) = 0. \end{aligned} \tag{116}$$

All $\hat\varphi(\hat x_h)$ belong to the $\hat H$-module $*$-algebra $\hat\Phi^e$. $\hat\Phi^e$ is generated by $\hat a^+_i, \hat a'^i$ and the sets of noncommutative coordinates $\hat x_1, \hat x_2, \ldots$ fulfilling (43), (71), (A.8); the latter relations provide our answer to the issues *(a)–(c)* mentioned in the introduction.

6.1. Relativistic QFT on Moyal–Minkowski space

Taking (51) as a twist (with Latin letters being replaced by Greek ones as indices) one obtains the Moyal–Minkowski noncommutative spacetime and the twisted Poincaré Hopf algebra $\hat H = \widehat{U\mathcal{P}}$ of [15, 47, 59][23],

$$\begin{aligned} &\hat\Delta(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu = \Delta(P_\mu),\\ &\hat\Delta(M_\omega) = M_\omega \otimes \mathbf{1} + \mathbf{1} \otimes M_\omega + [\omega, \theta]^{\mu\nu} P_\mu \otimes P_\nu \neq \Delta(M_\omega); \end{aligned}$$

here we have set $M_\omega := \omega^{\mu\nu} M_{\mu\nu}$. It is convenient to write formulae (71), (77) and (83) for generalized creation and annihilation operators:

$$\begin{aligned} &a^+_{\mathrm{p}} \star a^+_{\mathrm{q}} = \mathrm{e}^{-\mathrm{i}p\theta q} a^+_{\mathrm{q}} \star a^+_{\mathrm{p}}, & & \hat a^+_{\mathrm{p}}\hat a^+_{\mathrm{q}} = \mathrm{e}^{-\mathrm{i}p\theta q}\hat a^+_{\mathrm{q}}\hat a^+_{\mathrm{p}},\\ &a^{\mathrm{p}} \star a^{\mathrm{q}} = \mathrm{e}^{-\mathrm{i}p\theta q} a^{\mathrm{q}} \star a^{\mathrm{p}}, & & \hat a^{\mathrm{p}}\hat a^{\mathrm{q}} = \mathrm{e}^{-\mathrm{i}p\theta q}\hat a^{\mathrm{q}}\hat a^{\mathrm{p}},\\ &a^{\mathrm{p}} \star a^+_{\mathrm{q}} = \mathrm{e}^{\mathrm{i}p\theta q} a^+_{\mathrm{q}} \star a^{\mathrm{p}} + 2p^0\delta^3(\mathrm{p}-\mathrm{q}), & \Leftrightarrow\quad & \hat a^{\mathrm{p}}\hat a^+_{\mathrm{q}} = \mathrm{e}^{\mathrm{i}p\theta q}\hat a^+_{\mathrm{q}}\hat a^{\mathrm{p}} + 2p^0\delta^3(\mathrm{p}-\mathrm{q}),\\ &a^{\mathrm{p}} \star \mathrm{e}^{\mathrm{i}q\cdot x} = \mathrm{e}^{-\mathrm{i}p\theta q}\mathrm{e}^{\mathrm{i}q\cdot x} \star a^{\mathrm{p}}, \quad \text{and h.c.}, & & \hat a^{\mathrm{p}}\,\mathrm{e}^{\mathrm{i}q\cdot\hat x} = \mathrm{e}^{-\mathrm{i}p\theta q}\,\mathrm{e}^{\mathrm{i}q\cdot\hat x}\hat a^{\mathrm{p}}, \quad \text{and h.c.}; \end{aligned} \tag{117}$$

$$\check a^+_{\mathrm{p}} \equiv D^\sigma_{\mathcal{F}}\big(a^+_{\mathrm{p}}\big) = a^+_{\mathrm{p}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}p\theta\sigma(P)}, \qquad \check a^{\mathrm{p}} \equiv D^\sigma_{\mathcal{F}}(a^{\mathrm{p}}) = a^{\mathrm{p}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}p\theta\sigma(P)}, \tag{118}$$

[23] In section 4.4.1 of [47] this was formulated in terms of the dual Hopf algebra of $\hat H$.

$$\hat{a}^+_{p_1}\cdots\hat{a}^+_{p_n}\Psi_0 = a^+_{p_1}\star\cdots\star a^+_{p_n}\Psi_0 = \check{a}^+_{p_1}\cdots\check{a}^+_{p_n}\Psi_0 = \exp\left[-\frac{i}{2}\sum_{\substack{j,k=1\\ j<k}}^{n} p_j\theta p_k\right] a^+_{p_1}\cdots a^+_{p_n}\Psi_0 \tag{119}$$

where, according to definition (73), $\sigma(P_\mu) = \int d\mu(p)\, p_\mu a^+_p a^p$. By (119) generalized states differ from their undeformed counterparts only by multiplication by a phase factor. As $\check{a}^+_p\check{a}^p = a^+_p a^p$, $\sigma(P_\mu) = \int d\mu(p)\, p_\mu \check{a}^+_p\check{a}^p$, from (118) the inverse of $D^\sigma_{\mathcal{F}}$ is readily obtained. Formulae (116) become

$$\begin{aligned}
&\hat{\varphi}(\hat{x}) = \hat{\varphi}_i(\hat{x})\hat{a}'^i + \hat{a}^+_i\hat{\varphi}^{\hat{*}}_i(\hat{x}) = \int d\mu(p)\big[e^{-ip\cdot\hat{x}}\hat{a}'^p + \hat{a}^+_p e^{ip\cdot\hat{x}}\big],\\
&[\hat{\varphi}(\hat{x}),\hat{\varphi}(\hat{y})] = 2\int d\mu(p)\ \sin[p\cdot(\hat{x}-\hat{y})],\\
&(\hat{\Box}+m^2)\hat{\varphi}(\hat{x}) = 0.
\end{aligned}\tag{120}$$

The rhs$(120)_2$ is like the undeformed one.

Incidentally, by an explicit computation one can easily show the analogue of (85), i.e. the realization of the scalar product in (t-dependent) configuration space also holds on the Moyal–Minkowski space:

$$(s,v) = i\int d^3x\big[\psi^*_s\partial^0\psi_v - (\partial^0\psi^*_s)\psi_v\big] = i\int d^3x\left[\psi^{*_\star}_s\star\partial^0\psi_v - \left(\partial^0\psi^{*_\star}_s\right)\star\psi_v\right]; \tag{121}$$

one just needs to recall that $\beta = \mathbf{1}$ implies $*_\star = *$, replace the plane-wave expansions of ψ_s, ψ_v in (52) and note that integrating in d^3x gives a $\delta^3(h-k)$ and hence $h\theta k = 0$.

As seen, in terms of generalized wavefunctions, creation and annihilation operators relations (43), (71), (A.8) characterizing $\hat{\Phi}^e$ become (53), (117); the latter provide our answer on the Moyal–Minkowski space to the issues *(a)–(c)* mentioned in the introduction. It is remarkable that the free fields (115) and (117) coincide with the one found in formulae (37) and (46) of [32] (see also formulae (32) and (36) of [30]) imposing just the free field equation and Wightman axioms (modified only by the requirement of the *twisted* Poincaré covariance). The present construction shows that such a field is compatible with ordinary Bose/Fermi statistics—a point only briefly mentioned in [30, 32]. In [32] it has also been shown that the n-point functions of a (at least scalar) field theory, when expressed as functions of coordinates differences ξ, coincide with the undeformed ones. To a large extent this is due to (65). This result holds in time-ordered perturbation theory also for interacting fields with interaction $\varphi^{\star n}$, due to the translation invariance of the latter.

In our notation essentially all sets of relations recently appeared in [1, 2, 5, 9, 14, 16, 30, 32, 35, 44, 51, 58, 62] can be summarized as $\hat{x}^{h\hat{*}}_i = \hat{x}^h_i$, $\hat{\partial}_{x^h_i}{}^{\hat{*}} = -\hat{\partial}_{x^h_i}$ and

$$\begin{aligned}
&[\hat{x}^h_i,\hat{x}^k_j] = \mathbf{1}i\theta^{hk}(\eta_1+\eta_2\delta^i_j), \quad [\hat{\partial}_{x^h_i},\hat{x}^k_j] = \mathbf{1}\delta^k_h\delta^i_j, \quad [\hat{\partial}_{x^h_i},\hat{\partial}_{x^k_j}] = 0,\\
&\hat{a}^+_p\hat{a}^+_q = e^{-ip\tilde{\theta}q}\hat{a}^+_q\hat{a}^+_p, \quad \hat{a}^p\hat{a}^q = e^{-ip\tilde{\theta}q}\hat{a}^q\hat{a}^p, \quad \hat{a}^p\hat{a}^+_q = e^{ip\tilde{\theta}q}\hat{a}^+_q\hat{a}^p + 2p^0\delta^3(p-q),\\
&\hat{a}^p e^{iq\cdot\hat{x}} = e^{-i\eta_3 p\theta q}e^{iq\cdot\hat{x}}\hat{a}^p, \quad \hat{a}^+_p e^{iq\cdot\hat{x}} = e^{i\eta_3 p\theta q}e^{iq\cdot\hat{x}}\hat{a}^+_p,
\end{aligned}$$

the choices of the parameters $\tilde{\theta}^{\mu\nu}, \eta_1, \eta_2, \eta_3$ specifying the differences. Our relations (53) and (117), like (46) of [32] and (36) of [30], correspond to $\tilde{\theta} = \theta$, $\eta_1 = \eta_3 = 1$, $\eta_2 = 0$. In all other cases $\eta_3 = 0$, implying an answer to the issues *(a)–(c)* in the introduction different from the present one. The choice $\tilde{\theta} = 0$ gives the canonical (anti)commutation relations for the creation and annihilation operators and is assumed in most papers which do not twist the Poincaré covariance group, both in the operator (see e.g. [20] and [35]) and implicitly in the path-integral approach to quantization (see e.g. [26]), together with $\eta_1 = \eta_3 = 0$, $\eta_2 = 1$.

Chaichian and co-workers (see e.g. [16, 58]) do twist the Poincaré covariance group and all the spacetime coordinates adopting $\eta_1 = 1, \eta_2 = \eta_3 = 0$, but not the creation and annihilation operators ($\tilde{\theta} = 0$). Balachandran and co-workers (see [9] and e.g. the review [2]) adopt $\tilde{\theta} = -\theta$ and $\eta_1 = \eta_3 = 0, \eta_2 = 1$ (equivalently, they do not perform the $\star$-product between functions of different sets x, y of coordinates). In [1, 44, 51] $\tilde{\theta} = -\theta$ and $\eta_1 = 1, \eta_2 = \eta_3 = 0$ are adopted in scalar field theories respectively in 1+1 and in arbitrary dimension ([1] only with $\theta^{0i} = 0$). This is the only other choice leading to the 'local' free field commutation relation (120), and appears also in an alternative proposal (formulae (44) instead of (46)) contained in [32][24]. The other choices lead to a 'non-local' free field commutation relation. In [14] $\tilde{\theta} = \theta$ and $\eta_1 = \eta_2 = \eta_3 = 0$ are adopted (only creation and annihilation operators are twisted). In the prescription of [5] creation and annihilation operators are $\star$-multiplied as in the first three left equations of (117), which corresponds again to $\tilde{\theta} = \theta$ (it is not clear what the remaining commutation relations are, as they consider a different kind of $\star$-commutator $[\cdot,\cdot]_\star := [\cdot,\cdot] \circ \overline{\mathcal{F}}_{\triangleright}^{\otimes 2}$). In [62] $\eta_1 = 1, \eta_2 = \eta_3 = 0$ and $\tilde{\theta} = 0$ are considered (the creation and annihilation operators are not twisted).

A realization in the form (118) of generalized creation and annihilation operators fulfilling $(117)_2$ has appeared in [9]. It is also reminiscent of the Fock space realization [42] of the Zamolodchikov–Faddeev [63] algebra, which is generated by deformed creation and annihilation operators of scattering states of some completely integrable (1+1)-dimensional QFT.

Appendix

In this appendix we collect the proofs of several statements made in the previous sections and miscellaneous formulae are needed for that.

We start by writing in compact notation (5) and its consequences

$$\begin{aligned}
&\mathcal{F}_{12}\mathcal{F}_{(12)3} = \mathcal{F}_{23}\mathcal{F}_{1(23)}, \qquad &&\mathcal{F}_{(12)3}\mathcal{F}_{(123)4} = \mathcal{F}_{34}\mathcal{F}_{(12)(34)},\\
&\mathcal{F}_{1(23)}\mathcal{F}_{(123)4} = \mathcal{F}_{(23)4}\mathcal{F}_{1(234)}, \qquad &&\mathcal{F}_{12}\mathcal{F}_{(12)(34)} = \mathcal{F}_{2(34)}\mathcal{F}_{1(234)},
\end{aligned} \tag{A.1}$$

as well as the inverse of (5) and its consequences

$$\begin{aligned}
&\overline{\mathcal{F}}_{(12)3}\overline{\mathcal{F}}_{12} = \overline{\mathcal{F}}_{1(23)}\overline{\mathcal{F}}_{23}, \qquad &&\overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{(12)3} = \overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{34},\\
&\overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{1(23)} = \overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{(23)4}, \qquad &&\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{12} = \overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{2(34)},
\end{aligned} \tag{A.2}$$

obtained by applying Δ on the first, second, third tensor factor and taking into account the cocommutativity of Δ; the bracket encloses tensor factors obtained from one by application of Δ. To denote the decomposition of $\mathcal{F}_{(12)3}$ we use a Sweedler-type notation

$$\mathcal{F}_{(12)3} \equiv (\Delta \otimes \mathrm{id})(\mathcal{F}) = \sum_I \mathcal{F}^{(1)}_{(1)I} \otimes \mathcal{F}^{(1)}_{(2)I} \otimes \mathcal{F}^{(2)}_I,$$

and similarly for $\mathcal{F}_{1(23)}, \overline{\mathcal{F}}_{(12)3}, \ldots$. We denote as S_i the antipode on the ith tensor factor, and as τ_{ij} and m_{ij} respectively the flip and the multiplication of the ith, jth tensor factors.

Applying $m_{23} \circ S_3, m_{12} \circ S_2 \circ \tau_{12}$ to $(\mathrm{A.1})_1$, $m_{12} \circ S_1, m_{23} \circ S_2 \circ \tau_{23}$ to $(\mathrm{A.2})_1$ and recalling (6), (7) we respectively obtain the important relations (see e.g. also lemma 1 in [29])

$$\mathcal{F}^{-1} = \sum_I \mathcal{F}^{(1)}_{(1)I} \otimes \mathcal{F}^{(1)}_{(2)I} S\big(\mathcal{F}^{(2)}_I\big)\beta^{-1} = \mathcal{F}^{(2)}_{(1)I} S\big(\mathcal{F}^{(1)}_I\big) S(\beta^{-1}) \otimes \mathcal{F}^{(2)}_{(2)I}, \tag{A.3}$$

[24] However, in [30] we have put aside this alternative because scalar products cannot be expressed in terms of Wightman functions defined as vacuum expectation values of $\star$-products of fields.

$$\mathcal{F} = \sum_I \beta S\big(\overline{\mathcal{F}}_I^{(1)}\big)\overline{\mathcal{F}}_{(1)I}^{(2)} \otimes \overline{\mathcal{F}}_{(2)I}^{(2)} = \overline{\mathcal{F}}_{(1)I}^{(1)} \otimes S(\beta)S\big(\overline{\mathcal{F}}_I^{(2)}\big)\overline{\mathcal{F}}_{(1)I}^{(1)}. \tag{A.4}$$

Applying $m_{14}\circ m_{23}\circ S_3\circ S_4$ to (A.1)$_2$ and recalling (6), (7) we obtain $\Delta(\beta)[(S\otimes S)\mathcal{F}_{21}] = \mathcal{F}^{-1}(\beta\otimes\beta)$, implying the first equality in

$$\Delta(\beta) = \mathcal{F}^{-1}(\beta\otimes\beta)\big[(S\otimes S)\mathcal{F}_{21}^{-1}\big] = \mathcal{F}_{21}^{-1}(\beta\otimes\beta)[(S\otimes S)\mathcal{F}^{-1}]. \tag{A.5}$$

The second equality follows from the first and the cocommutativity of Δ.

Proof of (14), i.e. that (12) is an antihomomorphism for $\mathcal{A}_\star$.

$$\begin{aligned}
(a\star b)^{*_\star} &= \sum_I S(\beta)\triangleright\big[\big(\overline{\mathcal{F}}_I^{(1)}\triangleright a\big)\big(\overline{\mathcal{F}}_I^{(2)}\triangleright b\big)\big]^*\\
&= \sum_I S(\beta)\triangleright\big\{\big[S\big(\overline{\mathcal{F}}_I^{(2)}\big)^*\triangleright b^*\big]\big[S\big(\overline{\mathcal{F}}_I^{(1)}\big)^*\triangleright a^*\big]\big\}\\
&= \sum_I \big[S(\beta_{(1)})S\big(\mathcal{F}_I^{(2)}\big)S(\beta^{-1})\triangleright b^{*_\star}\big]\big[S(\beta_{(2)})S\big(\mathcal{F}_I^{(1)}\big)S(\beta^{-1})\triangleright a^{*_\star}\big]\\
&\overset{(A.5)}{=} \sum_I \big[\overline{\mathcal{F}}_I^{(1)}S(\beta)S(\beta^{-1})\triangleright b^{*_\star}\big]\big[\overline{\mathcal{F}}_I^{(2)}S(\beta)S(\beta^{-1})\triangleright a^{\hat{*}}\big] = b^{*_\star}\star a^{*_\star}.
\end{aligned}$$

□

Proof that $*_\star$ fulfils (11). Since $S(\beta)\beta \in \text{Centre}(H)[[\lambda]]$ and $\beta = \beta^{-1*}$, we find

$$\begin{aligned}
(g\triangleright a)^{*_\star} &\overset{(12)}{=} S(\beta)\triangleright(g\triangleright a)^* = S(\beta)\triangleright[S(g)]^*\triangleright a^* = S(\beta)[S(g)]^*S(\beta^{-1})\triangleright a^{*_\star}\\
&= \beta^{-1*}[S(g)]^*\beta^*\triangleright a^{*_\star} = [\beta S(g)\beta^{-1}]^*\triangleright a^{*_\star} = [\hat{S}(g)]^*\triangleright a^{*_\star}.
\end{aligned}$$

□

Proof of (18). Using (A.2) and $\mathcal{R}_{32} = \mathcal{F}_{23}\overline{\mathcal{F}}_{32}$ we find

$$\begin{aligned}
\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{34}\overline{\mathcal{F}}_{12}\mathcal{R}_{32} &= \overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{(12)3}\overline{\mathcal{F}}_{12}\mathcal{R}_{32} = \overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{1(23)}\overline{\mathcal{F}}_{23}\mathcal{R}_{32}\\
&= \overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{1(23)}\overline{\mathcal{F}}_{32} = \tau_{23}[\overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{1(23)}\overline{\mathcal{F}}_{23}] = \tau_{23}[\overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{(23)4}\overline{\mathcal{F}}_{23}]\\
&= \tau_{23}[\overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{2(34)}\overline{\mathcal{F}}_{34}] = \tau_{23}[\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{12}\overline{\mathcal{F}}_{34}]
\end{aligned}$$

whence

$$\begin{aligned}
\text{rhs}(18)_1 &= m_{12}m_{34}\left\{\big[\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{34}\overline{\mathcal{F}}_{12}\mathcal{R}_{32}\big]\triangleright^{\otimes 4}(a\otimes a'\otimes b\otimes b')\right\}\\
&= m_{12}m_{34}\tau_{23}\left\{\big[\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{12}\overline{\mathcal{F}}_{34}\big]\triangleright^{\otimes 4}(a\otimes b\otimes a'\otimes b')\right\}\\
&= m_{13}m_{24}\left\{\big[\overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{12}\overline{\mathcal{F}}_{34}\big]\triangleright^{\otimes 4}(a\otimes b\otimes a'\otimes b')\right\} = \text{lhs}(18)_1.
\end{aligned}$$

$(a\otimes_\star b)^{*_\star} = b_2^{*_\star}\star a_1^{*_\star}$ is just an application of (14) to $(\mathcal{A}\otimes\mathcal{B})_\star$; it takes the form (18)$_2$ upon using (18)$_1$. One can also directly check (18)$_2$ using (A.5). □

Proof of (30).

$$\begin{aligned}
\text{rhs}(30) &\overset{(26)}{=} \sum_I \sigma\big(\overline{\mathcal{F}}_{(1)I}^{(1)}\big)a\sigma\big[S\big(\overline{\mathcal{F}}_{(2)I}^{(1)}\big)\overline{\mathcal{F}}_I^{(2)}\big] = \sum_I \sigma\big(\overline{\mathcal{F}}_{(1)I}^{(1)}\big)a\sigma\big\{S\big[S\big(\overline{\mathcal{F}}_I^{(2)}\big)\overline{\mathcal{F}}_{(2)I}^{(1)}\big]\big\}\\
&\overset{(A.4)}{=} \sum_I \sigma\big(\mathcal{F}_I^{(1)}\big)a\sigma\big\{S\big[S\big(\beta^{-1}\big)\mathcal{F}_I^{(2)}\big]\big\} = \sum_I \sigma\big(\mathcal{F}_I^{(1)}\big)a\sigma\big\{S\big[\mathcal{F}_I^{(2)}\big]\beta^{-1}\big\} \overset{(28)}{=} D_{\mathcal{F}}^{\sigma}(a).
\end{aligned}$$

□

Proof of (31).

$$\begin{aligned}
\big[D_{\mathcal{F}}^{\sigma}(a)\big]^* &\overset{(30)}{=} \Big[\sum_I \big(\overline{\mathcal{F}}_I^{(1)}\triangleright a\big)\sigma\big(\overline{\mathcal{F}}_I^{(2)}\big)\Big]^* \overset{\mathcal{F}^{*_2}=\mathcal{F}^{-1}}{=} \sum_I \sigma\big(\mathcal{F}_I^{(2)}\big)\big[S\big(\mathcal{F}_I^{(1)}\big)\triangleright a^*\big]\\
&\overset{(26)}{=} \sum_I \big[\mathcal{F}_{(1)I}^{(2)}S\big(\mathcal{F}_I^{(1)}\big)\triangleright a^*\big]\sigma\big(\mathcal{F}_{(2)I}^{(2)}\big)\\
&\overset{(A.3)}{=} \sum_I \big[\overline{\mathcal{F}}_I^{(1)}S(\beta)\triangleright a^*\big]\sigma\big(\overline{\mathcal{F}}_I^{(2)}\big) \overset{(30)}{=} \big[D_{\mathcal{F}}^{\sigma}(a^{*_\star})\big].
\end{aligned}$$

□

Proof of (32).

$$\begin{aligned}
D^\sigma_{\mathcal{F}}(a)D^\sigma_{\mathcal{F}}(a') &\overset{(30)}{=} \sum_I \big(\overline{\mathcal{F}}^{(1)}_I \triangleright a\big)\sigma\big(\overline{\mathcal{F}}^{(2)}_I\big)\sum_{I'}\big(\overline{\mathcal{F}}^{(1)}_{I'} \triangleright a'\big)\sigma\big(\overline{\mathcal{F}}^{(2)}_{I'}\big)\\
&\overset{(26)}{=} \sum_{I,I'} \big(\overline{\mathcal{F}}^{(1)}_I \triangleright a\big)\big(\overline{\mathcal{F}}^{(2)}_{(1)I}\overline{\mathcal{F}}^{(1)}_{I'} \triangleright a'\big)\sigma\big(\overline{\mathcal{F}}^{(2)}_{(2)I}\overline{\mathcal{F}}^{(2)}_{I'}\big)\\
&\overset{(A.2)_1}{=} \sum_{I,I'} \big(\overline{\mathcal{F}}^{(1)}_{(1)I}\overline{\mathcal{F}}^{(1)}_{I'} \triangleright a\big)\big(\overline{\mathcal{F}}^{(1)}_{(2)I}\overline{\mathcal{F}}^{(2)}_{I'} \triangleright a'\big)\sigma\big(\overline{\mathcal{F}}^{(2)}_I\big)\\
&\overset{(10)}{=} \sum_{I,I'} \overline{\mathcal{F}}^{(1)}_I \triangleright \big[\big(\overline{\mathcal{F}}^{(1)}_{I'} \triangleright a\big)\big(\overline{\mathcal{F}}^{(2)}_{I'} \triangleright a'\big)\big]\sigma\big(\overline{\mathcal{F}}^{(2)}_I\big)\\
&\overset{(13)}{=} \sum_I \overline{\mathcal{F}}^{(1)}_I \triangleright \big[a \star a'\big]\sigma\big(\overline{\mathcal{F}}^{(2)}_I\big) \overset{(30)}{=} D^\sigma_{\mathcal{F}}(a\star a').
\end{aligned}$$

□

Proof that (35) defines a deforming map $D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}} : (\mathcal{A}\otimes\mathcal{B})_\star \mapsto (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$.

$$\begin{aligned}
g\hat\triangleright\big[D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}}(c)\big] &= \sum_I \hat\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(g^I_{(\hat1)}\big)\,\big[D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}}(c)\big]\,\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}\big[\hat S\big(g^I_{(\hat2)}\big)\big]\\
&\overset{(35)}{=} \mathcal{F}_\sigma \sum_{I,I'} \big[\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(g^I_{(\hat1)}\big)\big]\,\big(\overline{\mathcal{F}}^{(1)}_{I'} \triangleright c\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big[\overline{\mathcal{F}}^{(2)}_{I'}\hat S\big(g^I_{(\hat2)}\big)\big]\overline{\mathcal{F}}_\sigma\\
&\overset{(26)}{=} \mathcal{F}_\sigma \sum_{I,I'} \big[\big(g^I_{(\hat1)(1')}\overline{\mathcal{F}}^{(1)}_{I'}\big) \triangleright c\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big[g^I_{(\hat1)(2')}\overline{\mathcal{F}}^{(2)}_{I'}\hat S\big(g^I_{(\hat2)}\big)\big]\overline{\mathcal{F}}_\sigma\\
&\overset{(8)}{=} \mathcal{F}_\sigma \sum_{I,I'} \big[\big(\overline{\mathcal{F}}^{(1)}_{I'}g^I_{(\hat1)(\hat1')}\big) \triangleright c\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big[\overline{\mathcal{F}}^{(2)}_{I'}g^I_{(\hat1)(\hat2')}\hat S\big(g^I_{(\hat2)}\big)\big]\overline{\mathcal{F}}_\sigma\\
&= \mathcal{F}_\sigma \sum_{I,I'} \big[\big(\overline{\mathcal{F}}^{(1)}_{I'}g\big) \triangleright c\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}_{I'}\big)\overline{\mathcal{F}}_\sigma \overset{(35)}{=} \big[D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}}(g\triangleright c)\big],
\end{aligned}$$

proving the intertwining property (29) in the present case. (31) is proved as follows:

$$\begin{aligned}
\big[D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}}(c)\big]^{*\otimes*} &\overset{(35)}{=} \big[\mathcal{F}_\sigma\sum_I\big(\overline{\mathcal{F}}^{(1)}_I \triangleright c\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}_I\big)\overline{\mathcal{F}}_\sigma\big]^{*\otimes*}\\
&= \mathcal{F}_\sigma\sum_I \sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\mathcal{F}^{(2)}_I\big)\big[S\big(\mathcal{F}^{(1)}_I\big) \triangleright c^{*\otimes*}\big]\overline{\mathcal{F}}_\sigma\\
&\overset{(26)}{=} \mathcal{F}_\sigma\sum_I \big[\mathcal{F}^{(2)}_{(1)I}S\big(\mathcal{F}^{(1)}_I\big) \triangleright c^{*\otimes*}\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\mathcal{F}^{(2)}_{(2)I}\big)\overline{\mathcal{F}}_\sigma\\
&\overset{(A.3)}{=} \mathcal{F}_\sigma\sum_I \big[\overline{\mathcal{F}}^{(1)}_I S(\beta) \triangleright c^{*\otimes*}\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}\big)\overline{\mathcal{F}}_\sigma \overset{(35)}{=} \big[D^\sigma_{\mathcal{F}}(\widehat{c^{*\otimes*}})\big].
\end{aligned}$$

Finally, here is the proof of (32) in the present case:

$$\begin{aligned}
D^{\hat\sigma_{\mathcal{A}\otimes\mathcal{B}}}_{\mathcal{F}}(c\star c') &\overset{(35)}{=} \mathcal{F}_\sigma\sum_I \big[\overline{\mathcal{F}}^{(1)}_I \triangleright (c\star c')\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}_I\big)\overline{\mathcal{F}}_\sigma\\
&\overset{(13)}{=} \mathcal{F}_\sigma\sum_{I,I'} \big[\big(\overline{\mathcal{F}}^{(1)}_{(1)I}\overline{\mathcal{F}}^{(1')}_{I'}\big) \triangleright c\big]\big[\big(\overline{\mathcal{F}}^{(1)}_{(2)I}\overline{\mathcal{F}}^{(2')}_{I'}\big) \triangleright c'\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}_I\big)\overline{\mathcal{F}}_\sigma\\
&\overset{(5)}{=} \mathcal{F}_\sigma\sum_{I,I'} \big(\overline{\mathcal{F}}^{(1)}_I \triangleright c\big)\big[\big(\overline{\mathcal{F}}^{(2)}_{(1)I}\overline{\mathcal{F}}^{(1')}_{I'}\big) \triangleright c'\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}^{(2)}_{(2)I}\overline{\mathcal{F}}^{(2')}_{I'}\big)\overline{\mathcal{F}}_\sigma
\end{aligned}$$

$$\stackrel{(26)}{=} \mathcal{F}_\sigma \sum_{I,I'} \big(\overline{\mathcal{F}}_I^{(1)} \triangleright c\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}_I^{(2)}\big)\big[\overline{\mathcal{F}}_{I'}^{(1')} \triangleright c'\big]\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}_{I'}^{(2')}\big)\overline{\mathcal{F}}_\sigma$$

$$\stackrel{(35)}{=} \big[D_{\mathcal{F}}^{\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}}(c)\big]\big[D_{\mathcal{F}}^{\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}}(c')\big]. \qquad \square$$

Proof of (36).

$$\begin{aligned}
D_{\mathcal{F}}^{\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}}(\alpha\otimes\mathbf{1}) &\stackrel{(35)}{=} \mathcal{F}_\sigma \sum_I \big(\overline{\mathcal{F}}_I^{(1)} \triangleright \alpha\otimes\mathbf{1}\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}_I^{(2)}\big)\overline{\mathcal{F}}_\sigma \\
&= \sum_{I,I',I''} \big(\mathcal{F}_{(1')I'}^{(1')}\overline{\mathcal{F}}_I^{(1)} \triangleright \alpha\big)\sigma_{\mathcal{A}}\big(\mathcal{F}_{(2')I'}^{(1')}\overline{\mathcal{F}}_{(1)I}^{(2)}\overline{\mathcal{F}}_{I''}^{(1'')}\big)\otimes\sigma_{\mathcal{B}}\big(\mathcal{F}_{I'}^{(2')}\overline{\mathcal{F}}_{(2)I}^{(2)}\overline{\mathcal{F}}_{I''}^{(2'')}\big) \\
&\stackrel{(5)}{=} \sum_{I,I',I''} \big(\mathcal{F}_{(1')I'}^{(1')}\overline{\mathcal{F}}_{(1)I}^{(1)}\overline{\mathcal{F}}_{I''}^{(1'')} \triangleright \alpha\big)\sigma_{\mathcal{A}}\big(\mathcal{F}_{(2')I'}^{(1')}\overline{\mathcal{F}}_{(2)I}^{(1)}\overline{\mathcal{F}}_{I''}^{(2'')}\big)\otimes\sigma_{\mathcal{B}}\big(\mathcal{F}_{I'}^{(2')}\overline{\mathcal{F}}_I^{(2)}\big) \\
&= \sum_{I''} \big(\overline{\mathcal{F}}_{I''}^{(1'')} \triangleright \alpha\big)\sigma_{\mathcal{A}}\big(\overline{\mathcal{F}}_{I''}^{(2'')}\big)\otimes\mathbf{1} = D_{\mathcal{F}}^{\sigma_{\mathcal{A}}}(\alpha)\otimes\mathbf{1} = \check{\alpha}\otimes\mathbf{1}
\end{aligned}$$

$$\begin{aligned}
D_{\mathcal{F}}^{\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}}(\mathbf{1}\otimes b) &\stackrel{(35)}{=} (\overline{\mathcal{R}}\mathcal{F}_{21})_\sigma \sum_I \big(\mathbf{1}\otimes\overline{\mathcal{F}}_I^{(1)} \triangleright b\big)\sigma_{\mathcal{A}\otimes\mathcal{B}}\big(\overline{\mathcal{F}}_I^{(2)}\big)(\overline{\mathcal{F}}_{21}\mathcal{R})_\sigma \\
&= \overline{\mathcal{R}}_\sigma \sum_{I,I',I''} \big[\sigma_{\mathcal{A}}\big(\mathcal{F}_{I'}^{(2')}\overline{\mathcal{F}}_{(2)I}^{(2)}\overline{\mathcal{F}}_{I''}^{(2'')}\big)\otimes\big(\mathcal{F}_{(1')I'}^{(1')}\overline{\mathcal{F}}_I^{(1)} \triangleright b\big)\sigma_{\mathcal{B}}\big(\mathcal{F}_{(2')I'}^{(1')}\overline{\mathcal{F}}_{(1)I}^{(2)}\overline{\mathcal{F}}_{I''}^{(1'')}\big)\big]\mathcal{R}_\sigma \\
&\stackrel{(5)}{=} \overline{\mathcal{R}}_\sigma \sum_{I,I',I''} \big[\sigma_{\mathcal{A}}\big(\mathcal{F}_{I'}^{(2')}\overline{\mathcal{F}}_I^{(2)}\big)\otimes\big(\mathcal{F}_{(1')I'}^{(1')}\overline{\mathcal{F}}_{(1)I}^{(1)}\overline{\mathcal{F}}_{I''}^{(1'')} \triangleright b\big)\sigma_{\mathcal{B}}\big(\mathcal{F}_{(2')I'}^{(1')}\overline{\mathcal{F}}_{(2)I}^{(1)}\overline{\mathcal{F}}_{I''}^{(2'')}\big)\big]\mathcal{R}_\sigma \\
&= \overline{\mathcal{R}}_\sigma \sum_{I''} \big[\mathbf{1}\otimes\big(\overline{\mathcal{F}}_{I''}^{(1'')} \triangleright b\big)\sigma_{\mathcal{B}}\big(\overline{\mathcal{F}}_{I''}^{(2'')}\big)\big]\mathcal{R}_\sigma \\
&= \overline{\mathcal{R}}_\sigma\big[\mathbf{1}\otimes D_{\mathcal{F}}^{\sigma_{\mathcal{A}}}(b)\big]\mathcal{R}_\sigma = \overline{\mathcal{R}}_\sigma\big[\mathbf{1}\otimes\check{b}\big]\mathcal{R}_\sigma. \qquad \square
\end{aligned}$$

Proof of (40) and (43). Let $\partial'_i := S(\beta)\triangleright\partial_i = \tau^i_j(\beta)\partial_j$. $(39)_2$ implies $g\triangleright\partial'_i = \tau^i_j[\hat{S}(g)]\partial'_j$, which with $(39)_1$ proves the first line of (40). The second and third lines are proved as follows:

$$x^{a*_\star} \stackrel{(12)}{=} S(\beta)\triangleright x^{a*} = S(\beta)\triangleright x^a = \tau^k_a\big[S(\beta)\big]x^k,$$

$$\partial_a'^{*_\star} \stackrel{(12)}{=} S(\beta)\triangleright\big[S(\beta)\triangleright\partial_a\big]^* = S(\beta)S(\beta^*)\triangleright\partial_a^* = -S(\beta)S\big(\beta^{-1}\big)\triangleright\partial_a = -\tau^a_k\big(\beta^{-1}\big)\partial'_k,$$

$$x^a\star x^b \stackrel{(18)}{=} \sum_I\big[\mathcal{R}_I^{(2)}\triangleright x^b\big]\star\big[\mathcal{R}_I^{(1)}\triangleright x^a\big] = \sum_I \tau^h_b\big[\mathcal{R}_I^{(2)}\big]\tau^k_a\big[\mathcal{R}_I^{(1)}\big]x^h\star x^k = R^{kh}_{ab}x^h\star x^k,$$

$$\begin{aligned}
\partial'_a\star\partial'_b &\stackrel{(18)}{=} \sum_I\big[\mathcal{R}_I^{(2)}\triangleright\partial'_b\big]\star\big[\mathcal{R}_I^{(1)}\triangleright\partial'_a\big] = \sum_I \tau^b_h\big[\hat{S}\big(\mathcal{R}_I^{(2)}\big)\big]\tau^a_k\big[\hat{S}\big(\mathcal{R}_I^{(1)}\big)\big]\partial'_h\star\partial'_k \\
&\stackrel{(A.6)}{=} \sum_I \tau^b_h\big[\mathcal{R}_I^{(2)}\big]\tau^a_k\big[\mathcal{R}_I^{(1)}\big]\partial'_h\star\partial'_k = R^{ab}_{kh}\partial'_h\star\partial'_k,
\end{aligned}$$

$$\begin{aligned}
\partial'_a\star x^b &\stackrel{(13)}{=} \sum_I\big[\overline{\mathcal{F}}_I^{(1)}\triangleright\partial'_a\big]\star\big[\overline{\mathcal{F}}_I^{(2)}\triangleright x^b\big] = \sum_I \tau^a_k\big[\beta S\big(\overline{\mathcal{F}}_I^{(1)}\big)\big]\tau^h_b\big[\overline{\mathcal{F}}_I^{(2)}\big]\partial_k x^h \\
&= \sum_I \tau^a_k\big[\beta S\big(\overline{\mathcal{F}}_I^{(1)}\big)\big]\tau^h_b\big[\overline{\mathcal{F}}_I^{(2)}\big](\mathbf{1}\delta^h_k + x^h\partial_k) = \delta^a_b\mathbf{1} + \sum_I\big[\mathcal{R}_I^{(2)}\triangleright x^b\big]\star\big[\mathcal{R}_I^{(1)}\triangleright\partial'_a\big] \\
&= \delta^a_b\mathbf{1} + \sum_I \tau^h_b\big[\mathcal{R}_I^{(2)}\big]\tau^a_k\big[\hat{S}\big(\mathcal{R}_I^{(1)}\big)\big]x^h\star\partial'_k \stackrel{(A.6)}{=} \delta^a_b\mathbf{1} + \sum_I \tau^h_b\big[\mathcal{R}_I^{(1)}\big]\tau^a_k\big[\mathcal{R}_I^{(2)}\big]x^h\star\partial'_k \\
&= \delta^a_b\mathbf{1} + R^{ha}_{bk}x^h\star\partial'_k,
\end{aligned}$$

where we have used the following relation, valid for any triangular Hopf algebra:

$$(\hat{S} \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{21} = (\mathrm{id} \otimes \hat{S}^{-1})(\mathcal{R}). \tag{A.6}$$

The proof of (43) is completely analogous. □

Proof of (48).

$$\begin{aligned}\int_X \mathrm{d}\nu_j[\omega \star f(\mathrm{x}_j)] &\overset{(18)}{=} \int_X \mathrm{d}\nu_j \sum_I \left[\mathcal{R}_I^{(2)} \triangleright f(\mathrm{x}_j)\right] \star \left[\mathcal{R}_I^{(1)} \triangleright \omega\right] \\ &\overset{(44)}{=} \int_X \mathrm{d}\nu_j \left\{ f(\mathrm{x}_j) \star \sum_I \varepsilon\big(\mathcal{R}_I^{(2)}\big) \left[\mathcal{R}_I^{(1)} \triangleright \omega\right] \right\} \\ &\overset{(4)_2}{=} \int_X \mathrm{d}\nu_j\, f(\mathrm{x}_j) \star \omega \overset{(47)_2}{=} \omega \star \int_X \mathrm{d}\nu_j\, f(\mathrm{x}_j).\end{aligned}$$

□

Lemma 1. *Let $(\mathcal{M}, \rho)$ be a representation of H and $(\mathcal{M}^\vee, \rho^\vee)$ its contragredient, $\{\eta_i\}$ a basis of $\mathcal{M}$ and $\{\eta^i\}$ the contragredient one of $\mathcal{M}^\vee$. Then*

$$\eta_i \otimes \eta^i = \eta_i \otimes_\star \eta'^i \qquad (\textit{sum over } i) \qquad \eta'^i := S(\beta) \triangleright \eta^i \tag{A.7}$$

and this sum is H-invariant.

Proof. By definition $g \triangleright \eta_i = \rho_i^j(g)\eta_j$, $g \triangleright \eta^i = \rho^{\vee}{}^j_i(g)\eta^j = \rho^i_j[S(g)]\eta^j$. Therefore,

$$\begin{aligned}\eta_i \otimes_\star \eta'^i &= \sum_I \overline{\mathcal{F}}_I^{(1)} \triangleright \eta_i \otimes \overline{\mathcal{F}}_I^{(2)} S(\beta) \triangleright \eta^i = \sum_I \eta_j \rho_i^j\big(\overline{\mathcal{F}}_I^{(1)}\big) \otimes \rho_k^i\big[\beta S\big(\overline{\mathcal{F}}_I^{(2)}\big)\big]\eta^k \\ &= \sum_I \eta_j \otimes \rho_k^j\big[\overline{\mathcal{F}}_I^{(1)} \beta S\big(\overline{\mathcal{F}}_I^{(2)}\big)\big]\eta^k = \sum_{I,I'} \eta_j \otimes \rho_k^j\big[\overline{\mathcal{F}}_I^{(1)} \mathcal{F}_{I'}^{(1')} S\big(\overline{\mathcal{F}}_I^{(2)} \mathcal{F}_{I'}^{(2')}\big)\big]\eta^k = \eta_j \otimes \eta^j.\end{aligned}$$

□

The Commutation rules among $\hat{a}'^i$, $\hat{a}_j^+$, $\hat{\varphi}_i(\hat{\mathrm{x}})$, $\hat{\varphi}_i(\hat{\mathrm{y}})$, $\hat{\varphi}_i^{\hat{*}}(\hat{\mathrm{x}})$, $\hat{\varphi}_i^{\hat{*}}(\hat{\mathrm{y}})$, besides (71), are

$$\begin{aligned}&\hat{\varphi}_i(\hat{\mathrm{y}})\hat{\varphi}_j(\hat{\mathrm{x}}) = R_{ij}^{kh}\hat{\varphi}_h(\hat{\mathrm{x}})\hat{\varphi}_k(\hat{\mathrm{y}}), &&\hat{\varphi}_i^{\hat{*}}(\hat{\mathrm{y}})\hat{\varphi}_j^{\hat{*}}(\hat{\mathrm{x}}) = R_{kh}^{ij}\hat{\varphi}_h^{\hat{*}}(\hat{\mathrm{x}})\hat{\varphi}_k^{\hat{*}}(\hat{\mathrm{y}}),\\ &\hat{\varphi}_i^{\hat{*}}(\hat{\mathrm{y}})\hat{\varphi}_j(\hat{\mathrm{x}}) = R_{jk}^{hi}\hat{\varphi}_h(\hat{\mathrm{x}})\hat{\varphi}_k^{\hat{*}}(\hat{\mathrm{y}}), && \\ &\hat{\varphi}_i(\hat{\mathrm{x}})\hat{a}_j^+ = R_{ij}^{kh}\hat{a}_h^+\hat{\varphi}_k(\hat{\mathrm{x}}), &&\hat{\varphi}_i^{\hat{*}}(\hat{\mathrm{x}})\hat{a}_j^+ = R_{jk}^{hi}\hat{a}_h^+\hat{\varphi}_k^{\hat{*}}(\hat{\mathrm{x}}),\\ &\hat{a}'^i\hat{\varphi}_j(\hat{\mathrm{x}}) = R_{jk}^{hi}\hat{\varphi}_h(\hat{\mathrm{x}})\hat{a}'^k, &&\hat{a}'^i\hat{\varphi}_j^{\hat{*}}(\hat{\mathrm{x}}) = R_{kh}^{ij}\hat{\varphi}_h^{\hat{*}}(\hat{\mathrm{x}})\hat{a}'^k,\end{aligned} \tag{A.8}$$

where $\hat{\mathrm{x}}, \hat{\mathrm{y}} \in \{\hat{\mathrm{x}}_1, \hat{\mathrm{x}}_2, \ldots\}$ are any two sets of noncommutative coordinates fulfilling (43). Relations (A.8) also hold if $\hat{\mathrm{y}} = \hat{\mathrm{x}}$, and are a straightforward consequence of (18). □

Proof of (74). Any s can be expressed in the form $s = \sum_J c_s^J \Psi_0$, with some $c_s^J \in \mathcal{A}^{\pm}[[\lambda]]$. Using the identities $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$, $\varepsilon \circ S = \varepsilon$ we find

$$\begin{aligned}g \triangleright s &= \sum_J g \triangleright \big(c_s^J \Psi_0\big) \overset{(69)}{=} \sum_{J,I} \big(g_{(1)}^I \triangleright c_s^J\big) g_{(2)}^I \triangleright \Psi_0 \\ &= \sum_{J,I} \big(g_{(1)}^I \triangleright c_s^J\big)\varepsilon\big(g_{(2)}^I\big)\Psi_0 = \sum_J \big(g \triangleright c_s^J\big)\Psi_0 \\ &\overset{(26)}{=} \sum_{J,I} \sigma\big(g_{(1)}^I\big)c_s^J \sigma\left[S\left(g_{(2)}^I\right)\right]\Psi_0 = \sigma\left(g_{(1)}^I\right)c_s^J \varepsilon\left[S\left(g_{(2)}^I\right)\right]\Psi_0 = \sum_J \sigma(g)c_s^J\Psi_0 = \sigma(g)s.\end{aligned}$$

□

Proof of (76). Using the identities $(\mathrm{id}\otimes\hat{\varepsilon})\circ\hat{\Delta}=\mathrm{id}$ and $\cdot\circ(\hat{S}\otimes\mathrm{id})\circ\hat{\Delta}=\hat{\varepsilon}$ ($\cdot$ stands for the product in $\hat{H}$), we find

$$g\hat{\triangleright}(cs)\stackrel{(75)}{=}\sigma(g)cs=\sum_I\sigma\left(g^I_{(1)}\right)c\varepsilon\left(g^I_{(2)}\right)s=\sum_I\sigma\left(g^I_{(1)}\right)c\sigma\left[\hat{S}\left(g^I_{(2)}\right)g^I_{(3)}\right]s$$
$$\stackrel{(27)}{=}\sum_I\left(g^I_{(1)}\hat{\triangleright}c\right)\sigma\left(g^I_{(2)}\right)s\stackrel{(75)}{=}\sum_I(g^I_{(\hat{1})}\hat{\triangleright}c)g^I_{(\hat{2})}\hat{\triangleright}s.$$

□

Proof of (81). Let $v_1,v_2\in\mathcal{A}^{\pm}[[\lambda]]$ be such that $s_h=v_h\Psi_0$; then

$$\begin{aligned}\langle cs_1,s_2\rangle&=\langle cv_1\Psi_0,v_2\Psi_0\rangle=\langle\Psi_0,(cv_1)^*v_2\Psi_0\rangle\\&\stackrel{(82)}{=}\langle\Psi_0,(cv_1)^{*_\star}\star v_2\Psi_0\rangle\stackrel{(14)}{=}\langle\Psi_0,v_1^{*_\star}\star c^{*_\star}\star v_2\Psi_0\rangle\\&\stackrel{(82)}{=}\langle\Psi_0,v_1^*(c^{*_\star}\star v_2)\Psi_0\rangle=\langle v_1\Psi_0,c^{*_\star}\star v_2\Psi_0\rangle=\langle s_1,c^{*_\star}\star s_2\rangle.\end{aligned}$$

□

Proof of the commutativity of the diagram shown in figure 1. We start by noting that

$$\begin{aligned}\frac{1}{\sqrt{n!}}\varphi^*(\mathrm{x}_1)\cdots\varphi^*(\mathrm{x}_n)\Psi_0&=a^+_{i_1}\cdots a^+_{i_n}\Psi_0\frac{1}{\sqrt{n!}}\varphi^*_{i_n}(\mathrm{x}_n)\cdots\varphi^*_{i_1}(\mathrm{x}_1)\\&=|n_1,n_2\ldots\rangle\frac{1}{N}\varphi^*_{i_n}(\mathrm{x}_n)\ldots\varphi^*_{i_1}(\mathrm{x}_1)\\&=e^{\pm}_{i_1,\ldots,i_n}\frac{1}{N}\varphi^*_{i_n}(\mathrm{x}_n)\cdots\varphi^*_{i_1}(\mathrm{x}_1)=(e_{j_1}\otimes\cdots\otimes e_{j_n})\mathcal{P}^{n\,j_1\ldots j_n}_{\pm i_1\ldots i_n}\varphi^*_{i_n}(\mathrm{x}_n)\ldots\varphi^*_{i_1}(\mathrm{x}_1).\end{aligned}$$

As an example, we prove that $\pi^n_\pm=\kappa^{\otimes n}\circ\mathcal{P}^n_\pm$ and $\Pi^n_\pm=\mathcal{P}^n_\pm\circ(\kappa^{\otimes n})^{-1}$. In fact, for any $s=s^{j_1,\ldots,j_n}(e_{j_1}\otimes\cdots\otimes e_{j_n})\in\mathcal{H}^{\otimes n}$, $\psi=\psi^{j_1,\ldots,j_n}(\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_n})\in\mathcal{X}^{\otimes n}$ we find

$$\begin{aligned}\left[\pi^n_\pm(s)\right](\mathrm{x}_1,\ldots,\mathrm{x}_n)&=\frac{1}{\sqrt{n!}}\left\langle\left[\varphi^*(\mathrm{x}_1)\cdots\varphi^*(\mathrm{x}_n)\Psi_0\right],s\right\rangle=\varphi_{j_1}(\mathrm{x}_1)\cdots\varphi_{j_n}(\mathrm{x}_n)\mathcal{P}^{n\,j_1\ldots j_n}_{\pm i_1\ldots i_n}s^{i_1,\ldots,i_n}\\&=\mathcal{P}^{n\,j_1\ldots j_n}_{\pm i_1\ldots i_n}s^{i_1,\ldots,i_n}[\kappa^{\otimes n}(e_{j_1}\otimes\ldots\otimes e_{j_n})](\mathrm{x}_1,\ldots,\mathrm{x}_n)=\{\kappa^{\otimes n}[\mathcal{P}^n_\pm(s)]\}(\mathrm{x}_1,\ldots,\mathrm{x}_n)\\\Pi^n_\pm(\psi)&=\int_X\mathrm{d}\nu(\mathrm{x}_1)\cdots\int_X\mathrm{d}\nu(\mathrm{x}_n)\frac{1}{\sqrt{n!}}\varphi^*(\mathrm{x}_1)\cdots\varphi^*(\mathrm{x}_n)\Psi_0\,\psi(\mathrm{x}_1,\cdots,\mathrm{x}_n)\\&=(e_{j_1}\otimes\cdots\otimes e_{j_n})\mathcal{P}^{n\,j_1\ldots j_n}_{\pm i_1\ldots i_n}\int_X\mathrm{d}\nu(\mathrm{x}_1)\\&\quad\cdots\int_X\mathrm{d}\nu(\mathrm{x}_n)\,\varphi^*_{i_n}(\mathrm{x}_n)\cdots\varphi^*_{i_1}(\mathrm{x}_1)\varphi_{h_1}(\mathrm{x}_1)\ldots\varphi_{h_n}(\mathrm{x}_n)\psi^{h_1,\ldots,h_n}\\&=(e_{j_1}\otimes\cdots\otimes e_{j_n})\mathcal{P}^{n\,j_1\ldots j_n}_{\pm i_1\ldots i_n}\psi^{i_1,\ldots,i_n}=\mathcal{P}^n_\pm(e_{i_1}\otimes\cdots\otimes e_{i_n})\psi^{i_1,\ldots,i_n}\\&=\mathcal{P}^n_\pm[\left(\kappa^{\otimes n}\right)^{-1}(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n})]\psi^{i_1,\ldots,i_n}=\mathcal{P}^n_\pm[(\kappa^{\otimes n})^{-1}(\psi)].\end{aligned}$$

□

Proof. Proof of (99): By (83) $e'^{\pm}_{i_1,\ldots,i_n}=\overline{F}^{n\,j_1\ldots j_n}_{\;i_1\ldots i_n}e^{\pm}_{j_1,\ldots,j_n}$, whence

$$\mathrm{lhs}(99)\stackrel{(98)}{=}\overline{F}^{n\,j_1\ldots j_n}_{\;i_1\ldots i_n}F^{n\,l_1\ldots l_n}_{\;(j_1\ldots j_n]}\hat{\varphi}_{l_1}(\hat{\mathrm{x}}_1)\ldots\hat{\varphi}_{l_n}(\hat{\mathrm{x}}_n)=\mathcal{P}^{n,F\,l_1\ldots l_n}_{\pm\;\;i_1\ldots i_n}\hat{\varphi}_{j_1}(\hat{\mathrm{x}}_1)\ldots\hat{\varphi}_{j_n}(\hat{\mathrm{x}}_n)$$

proving the first equality in (99); to prove the second one can use either (38) or (88) (together with their generalizations to $n>2$). □

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